

# Calculus (Problem Set)

Srijan Chattopadhyay

July 21, 2025

1.  $X_0, X_1, \alpha \in (0, 1)$ .  $X_{n+1} = \alpha X_n + (1 - \alpha)X_{n-1} \forall n \geq 1$ . Prove that,  $\{X_n\}$  converges and find the limit.
2. If  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $f(x + y) = f(x) + f(y)$  and  $f$  is monotonically increasing. Then, prove that  $\exists a$  such that  $f(x) = ax \forall x \in \mathbb{R}$ .
3. **Algorithm to Compute  $\sqrt{\alpha}, \alpha > 0$ :** Start with any  $x_1 > \sqrt{\alpha}$ . Define  $x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n})$ . Show that  $\{x_n\}$  is monotonically decreasing and hence find the limit of  $\{x_n\}$ .
4. Prove that for any prime  $p \geq 3$ , the number  $\binom{2p-1}{p-1} - 1$  is divisible by  $p^2$ .
5. For sets  $A, B$ , let  $f : A \rightarrow B$  and  $g : B \rightarrow A$  be functions such that  $f(g(x)) = x$  for each  $x$ . Prove that :
  - $f$  need not be one one
  - $f$  must be onto
  - $g$  must be one one
  - $g$  need not to be onto.
6. Let  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . Then, prove that  $\frac{x_1 y_n + x_2 y_{n-1} + \dots + x_n y_1}{n} \rightarrow xy$ .
7. Let  $0 < a \leq x_1 \leq x_2 \leq b$ . Define  $x_n = \sqrt{x_{n-1} x_{n-2}}$  for  $n \geq 3$ . Show that  $a \leq x_n \leq b$  and  $|x_{n+1} - x_n| \leq \frac{b}{a+b} |x_n - x_{n-1}|$  for  $n \geq 2$ . Prove  $\{x_n\}$  is convergent.
8. Let  $0 < y_1 < x_1$ . Define  $x_{n+1} = \frac{x_n + y_n}{2}$  and  $y_{n+1} = \sqrt{x_n y_n}$ , for  $n \in \mathbb{N}$ . Prove that:
  - $\{y_n\}$  is increasing and bounded above while  $\{x_n\}$  is decreasing and bounded below.
  - $0 < x_{n+1} - y_{n+1} < 2^{-n}(x_1 - y_1)$  for  $n \in \mathbb{N}$ .
  - Prove that  $x_n$  and  $y_n$  converge to the same limit.
9. **Euler's Constant:** Let

$$\gamma_n = \sum_{k=1}^n \frac{1}{k} - \log n = \sum_{k=1}^n \frac{1}{k} - \int_1^n t^{-1} dt$$

- Show that  $\gamma_n$  is a decreasing sequence.
  - Show that  $0 < \gamma_n \leq 1$  for all  $n$ .
  - $\lim \gamma_n$  exists and is denoted by  $\gamma$ . The real number  $\gamma$  is called the Euler's constant.
10. Evaluate  $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n \cdot 3^m + m \cdot 3^n)}$
  11. Find the value of  $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{3^i 3^j 3^k}$ , where the sum is taken over  $i, j, k$  such that no two of them can be equal.
  12. Let  $S_n, n = 1, 2, 3, \dots$  be the sum of infinite geometric series, whose first term is  $n$  and the common ratio is  $\frac{1}{n+1}$ . Evaluate

$$\lim_{n \rightarrow \infty} \frac{S_1 S_n + S_2 S_{n-1} + S_3 S_{n-2} + \dots + S_n S_1}{S_1^2 + S_2^2 + \dots + S_n^2}$$

13.  $t_n = \frac{n^5+n^3}{n^4+n^2+1}$ ,  $S_r = \sum_{n=1}^r t_n$ . Find an explicit expression of  $S_r$ .
14. Find the following limits (if exists):
- $\lim_{x \rightarrow 0} \frac{(x - \sin x)}{x^3}$
  - $\lim_{x \rightarrow 0} \frac{(e^x - 1 - x)}{x^2}$
  - $\lim_{x \rightarrow 0} \left( \frac{a^x + b^x + c^x}{3} \right)^{1/x}$ ,  $a, b, c > 0$
  - $\lim_{x \rightarrow 0} ((x+a)(x+b)(x+c))^{1/3} - x$
15.  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $|f(x) - f(y)| \leq \lambda|x - y|$  for all  $x, y \in \mathbb{R}$  for some  $\lambda > 0$ . Prove that  $f$  is continuous (This is called **Lipschitz continuous**).
16. Find all  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $|f(x) - f(y)| \leq \lambda(x - y)^2$  for all  $x, y \in \mathbb{R}$  for some  $\lambda > 0$ .
17.  $f(x) = x \sin(1/x)$  if  $x \neq 0$  and  $= 0$  if  $x = 0$ . Prove that  $x$  is continuous.
18. Can you find minimum value of  $r$  such that if  $f(x) = x^r \sin(1/x)$  if  $x \neq 0$  and  $= 0$  if  $x = 0$ , then  $f$  is at least once differentiable?

19.

$$f(x) = \begin{cases} 0, & \text{for } x \in Q \\ 1, & \text{for } x \in Q^c \end{cases}$$

$Q$  is the set of all rational numbers. Find all continuity points of  $f$  (This function is called Dirichlet Function).

20. Can there exist any continuous  $f : \mathbb{R} \rightarrow \mathbb{R}$  which is rational on irrational points and irrational on rational points?
21.  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $\exists \alpha \in (0, 1)$  such that  $|f(x) - f(y)| \leq \alpha|x - y|$ . Prove that  $f$  has a unique fixed point. (A point  $x$  is called a fixed point of  $f$  is  $f(x) = x$ ).
22.  $f, g : [0, 1] \rightarrow [0, 1]$  are continuous functions such that  $f(g(x)) = g(f(x))$  for all  $x \in \mathbb{R}$ . Show that  $\exists c \in [0, 1]$  such that  $f(c) = g(c)$ .
23. Suppose that a function  $f$  is continuous on the interval  $[a, b]$  and differentiable on  $(a, b)$ . If the graph of  $f$  is not a line segment, prove that there exists a point  $c \in (a, b)$  such that  $|f'(c)| > \left| \frac{f(b) - f(a)}{b - a} \right|$ .
24. Let  $f$  be a twice differentiable function on the open interval  $(-1, 1)$  such that  $f(0) = 1$ . Suppose  $f$  also satisfies  $f(x) \geq 0$ ,  $f'(x) \leq 0$  and  $f''(x) \leq f(x)$ , for all  $x \geq 0$ . Show that  $f'(0) \geq -\sqrt{2}$ .
25. Find the limit

$$\lim_{x \rightarrow 0} \frac{\sin \tan x - \tan \sin x}{\arcsin \arctan x - \arctan \arcsin x}$$

26. Calculate the 100th derivative of the function

$$\frac{x^2 + 1}{x^3 - x}$$

27. Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a bijection of the positive integers. Prove that at least one of the following limits is true:

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{n + f(n)} = \infty; \quad \lim_{N \rightarrow \infty} \sum_{n=1}^N \left( \frac{1}{n} - \frac{1}{f(n)} \right) = \infty.$$

28. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function such that  $f'(x) > f(x) > 0$  for all real numbers  $x$ . Show that  $f(8) > 2024f(0)$ .

29. Evaluate

$$\frac{1/2}{1 + \sqrt{2}} + \frac{1/4}{1 + \sqrt[4]{2}} + \frac{1/8}{1 + \sqrt[8]{2}} + \frac{1/16}{1 + \sqrt[16]{2}} + \dots$$

30. Prove that for any function  $f : \mathbb{Q} \rightarrow \mathbb{Z}$ , there exist  $a, b, c \in \mathbb{Q}$  such that  $a < b < c$ ,  $f(b) \geq f(a)$  and  $f(b) \geq f(c)$ .

31. Define the sequence  $x_1, x_2, \dots$  by the initial terms  $x_1 = 2, x_2 = 4$ , and the recurrence relation

$$x_{n+2} = 3x_{n+1} - 2x_n + \frac{2^n}{x_n} \quad \text{for } n \geq 1.$$

Prove that  $\lim_{n \rightarrow \infty} \frac{x_n}{2^n}$  exists and satisfies

$$\frac{1 + \sqrt{3}}{2} \leq \lim_{n \rightarrow \infty} \frac{x_n}{2^n} \leq \frac{3}{2}.$$

32. Let  $x_1 = 2021$ ,  $x_n^2 - 2(x_n + 1)x_{n+1} + 2021 = 0$  ( $n \geq 1$ ). Prove that the sequence  $x_n$  converges. Find the limit  $\lim_{n \rightarrow \infty} x_n$ .

33. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuously differentiable function. Prove that

$$\left| f(1) - \int_0^1 f(x) dx \right| \leq \frac{1}{2} \max_{x \in [0,1]} |f'(x)|.$$

34. Let  $0 < a < 1$ . Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  continuous at  $x = 0$  such that  $f(x) + f(ax) = x$ ,  $\forall x \in \mathbb{R}$ .

35. How many ordered pairs of real numbers  $(a, b)$  satisfy equality  $\lim_{x \rightarrow 0} \frac{\sin^2 x}{e^{ax} - 2bx - 1} = \frac{1}{2}$ ?

36. For a given integer  $n \geq 1$ , let  $f : [0, 1] \rightarrow \mathbb{R}$  be a non-decreasing function. Prove that

$$\int_0^1 f(x) dx \leq (n+1) \int_0^1 x^n f(x) dx.$$

37. For  $n = 1, 2, \dots$  let

$$S_n = \log \left( \sqrt[n^2]{1^1 \cdot 2^2 \cdot \dots \cdot n^n} \right) - \log(\sqrt{n}),$$

where  $\log$  denotes the natural logarithm. Find  $\lim_{n \rightarrow \infty} S_n$ .

38. Let  $V$  be the set of all continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$ , differentiable on  $(0, 1)$ , with the property that  $f(0) = 0$  and  $f(1) = 1$ . Determine all  $\alpha \in \mathbb{R}$  such that for every  $f \in V$ , there exists some  $\xi \in (0, 1)$  such that

$$f(\xi) + \alpha = f'(\xi)$$

39. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function whose second derivative is continuous. Suppose that  $f$  and  $f''$  are bounded. Show that  $f'$  is also bounded.

40. Calculate the exact value of the series  $\sum_{n=2}^{\infty} \log(n^3 + 1) - \log(n^3 - 1)$  and provide justification.

41. Find the limit

$$\lim_{n \rightarrow \infty} \left( \frac{\left(1 + \frac{1}{n}\right)^n}{e} \right)^n.$$

42. A real-valued function  $f$  defined in  $(a, b)$  is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever  $a < x < b$ ,  $a < y < b$ ,  $0 < \lambda < 1$ .

- Prove that every convex function is continuous.

- Prove that every increasing convex function of a convex function is convex.
- If  $f$  is convex in  $(a, b)$  and if  $a < s < t < u < b$ , show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

- Let  $f : (a, b) \rightarrow \mathbb{R}$  such that  $f''$  exists on  $(a, b)$ . Then  $f$  is convex iff  $f''(x) > 0 \forall x \in (a, b)$ .
43.  $f : [0, 1] \rightarrow \mathbb{R}$  differentiable on  $(0, 1)$ , such that  $\exists a, b \in (0, 1)$  such that  $\int_0^a f(x)dx = 0$  and  $\int_b^1 f(x)dx = 0$ . Also,  $|f'(x)| \leq M \forall x \in (0, 1)$ . Prove that  $|\int_0^1 f(x)dx| \leq \frac{(1-a+b)M}{4}$ .
- First prove that,  $|\int_0^1 f(x)dx| \leq \frac{M(1-a)}{2}$  [Hint: Use MVT and a substitution to use  $\int_0^a f(x)dx = 0$ , Think how you can transform the interval  $(0, 1)$  to  $(0, a)$ ].
  - Then, prove that  $|\int_0^1 f(x)dx| \leq \frac{Mb}{2}$  [Hint: Again use MVT and a substitution to use  $\int_b^1 f(x)dx = 0$ , Think how you can transform the interval  $(0, 1)$  to  $(b, 1)$ ]
44. Show that there doesn't exist any increasing differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}^+$  such that  $f(f(x)) = f'(x)$
45. Determine all continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$  that satisfy  $\int_0^1 f(x)(1 - f(x))dx = \frac{1}{12}$
46. Let  $f(x) = \int_0^1 |t - x|tdt \forall x \in \mathbb{R}$ . Sketch the graph of  $f(x)$ . What is the minimum value of  $f(x)$ .
47.  $f(x) = \int_x^{x+1} \sin(u^2)du$ , find  $\lim_{x \rightarrow \infty} f(x)$ , if it exists.
48. The sequence  $\{q_n(x)\}$  of polynomials is defined by

$$q_1(x) = 1 + x, \quad q_2(x) = 1 + 2x$$

and for  $m \geq 1$  by

$$\begin{aligned} q_{2m+1}(x) &= q_{2m}(x) + (m+1)xq_{2m-1}(x), \\ q_{2m+2}(x) &= q_{2m+1}(x) + (m+1)xq_{2m}(x). \end{aligned}$$

Let  $x_n$  be the largest real solution of  $q_n(x) = 0$ . Prove that

- the sequence  $\{x_n\}$  is increasing.
  - $x_{2m+2} > -\frac{1}{m+1}$  for  $m \geq 1$ .
  - the sequence  $\{x_n\}$  converges to 0.
49. Let the positive integers  $a, b, c$  be such that  $a \geq b \geq c$  and  $(a^x - b^x - c^x)(x - 2) > 0$  for all  $x \neq 2$ . Show that  $a, b, c$  are sides of a right-angled triangle.
50. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a differentiable function such that  $f'$  is continuous and

$$f(0) = 0, \quad f(1) = 1.$$

- Show that there exists  $x_1$  in  $(0, 1)$  such that

$$\frac{1}{f'(x_1)} = 1.$$

- Show that there exist distinct  $x_1, x_2$  in  $(0, 1)$  such that

$$\frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} = 2.$$

- Show that for a positive integer  $n$ , there exist distinct  $x_1, x_2, \dots, x_n$  in  $(0, 1)$  such that

$$\sum_{i=1}^n \frac{1}{f'(x_i)} = n.$$

51. Define a sequence  $(a_n)$  for  $n \geq 1$  by  $a_1 = 2$  and  $a_{n+1} = a_n^{1+n^{-3/2}}$ . Is  $\lim_{n \rightarrow \infty} a_n < \infty$ ?
52. Let  $P(x) = x^{100} + 20x^{99} + 198x^{98} + a_9x^{97} + \dots + a_1x + 1$  be a polynomial where the  $a_i$  ( $1 \leq i \leq 97$ ) are real numbers. Prove that the equation  $P(x) = 0$  has at least one nonreal root.
53. Find  $\lim_{x \rightarrow \infty} \left( (2x)^{1+\frac{1}{2x}} - x^{1+\frac{1}{x}} - x \right)$
54. Find  $\sum_{k=1}^{\infty} \frac{k^2-2}{(k+2)!}$ .
55. A sequence  $(a_n)$  is defined by  $a_0 = -1, a_1 = 0$ , and  $a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1$  for all positive integers  $n$ . Find  $a_{100}$ .
56. Suppose that  $f : [-1, 1] \rightarrow \mathbb{R}$  is continuous and satisfies

$$\left( \int_{-1}^1 e^x f(x) dx \right)^2 \geq \left( \int_{-1}^1 f(x) dx \right) \left( \int_{-1}^1 e^{2x} f(x) dx \right).$$

Prove that there exists a point  $c \in (-1, 1)$  such that  $f(c) = 0$ . [You can assume CS inequality for integrals.]

57. Let  $f$  and  $g$  be two continuous, distinct functions from  $[0, 1] \rightarrow (0, +\infty)$  such that

$$\int_0^1 f(x) dx = \int_0^1 g(x) dx$$

Let

$$y_n = \int_0^1 \frac{f^{n+1}(x)}{g^n(x)} dx, \text{ for } n \geq 0, \text{ natural.}$$

Prove that  $(y_n)$  is an increasing and divergent sequence.

58. Consider a function  $f : [0, 1] \rightarrow [0, 1]$  satisfying the following property  $|f(x) - f(y)| < |x - y|$  for all  $x, y \in X, x \neq y$ . Show that  $f$  has a fixed point. Is the fixed point unique? [Hint : Define  $d(x) = |x - f(x)|$ . Suppose  $\inf_x d(x) \geq \epsilon > 0$ . Assume  $\exists x_0$  such that the infimum is attained. (Can you prove this? You can skip if you can't!). Then, use the property of  $f$  to arrive at a contradiction.]
59. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function given by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 5x - 6 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Find continuity points of  $f$ .

60. Show that the polynomial equation with real coefficients  $a_n x^n + a_{n-1} x^{n-1} + \dots + a_3 x^3 + x^2 + x + 1 = 0$  cannot have all real roots.
61. Assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous, one-to-one function. If there exists a positive integer  $n$  such that  $f^n(x) = x$ , for every  $x \in \mathbb{R}$ , then prove that either  $f(x) = x$  or  $f^2(x) = x$ . (Note that  $f^n(x) = f(f^{n-1}(x))$ .)
62. Consider  $f(x) = \frac{x^3}{6} + \frac{x^2}{2} + \frac{x}{3} + 1$ . Prove that  $f(x)$  is an integer whenever  $x$  is an integer. Determine with justification, conditions on real numbers  $a, b, c$  and  $d$  so that  $ax^3 + bx^2 + cx + d$  is an integer for all integers  $x$ .
63. (a) Show that there does not exist a function  $f : (0, \infty) \rightarrow (0, \infty)$  such that

$$f''(x) \leq 0 \text{ for all } x \text{ and } f'(x_0) < 0 \text{ for some } x_0.$$

- (b) Let  $k \geq 2$  be any integer. Show that there does not exist an infinitely differentiable function  $f : (0, \infty) \rightarrow (0, \infty)$  such that

$$f^{(k)}(x) \leq 0 \text{ for all } x \text{ and } f^{(k-1)}(x_0) < 0 \text{ for some } x_0.$$

Here,  $f^{(k)}$  denotes the  $k^{\text{th}}$  derivative of  $f$ .

64. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. Can you put appropriate conditions on  $f$ , so that  $f$  restricted to  $(a, b)$ ,  $-\infty < a < b < \infty$  can't attain its maximum and minimum inside  $(a, b)$ ?
65. Let  $a, b \in \mathbb{Z}$  and  $b > 0$ . Let  $f : [-1, 1] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that

- (a)  $f$  is continuous if and only if  $a > 0$ .
  - (b)  $f'(0)$  exists if and only if  $a > 1$ .
  - (c)  $f'$  is bounded if and only if  $a \geq 1 + b$ .
  - (d)  $f'$  is continuous if and only if  $a > 1 + b$ .
66. Let  $f : (0, 1] \rightarrow \mathbb{R}$  be a differentiable function with  $f'$  bounded on  $(0, 1]$ . Define

$$a_n = f\left(\frac{1}{n}\right), \quad n \geq 1.$$

Show that  $\{a_n\}$  is a convergent sequence.

67. Let  $f$  be a thrice differentiable function on  $(0, 1)$  such that  $f(x) \geq 0$  for all  $x \in (0, 1)$ . If  $f(x) = 0$  for at least two values of  $x \in (0, 1)$ , prove that  $f'''(c) = 0$  for some  $c \in (0, 1)$ .
68. Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function and that  $f(x+1) - f(x)$  converges to 0 as  $x \rightarrow \infty$ . Then show that

$$\frac{f(x)}{x} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

69. Let  $f$  be continuous, non-negative, and assume that

$$\int_0^\infty f(x) dx < \infty.$$

Then show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n x f(x) dx = 0.$$

70. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a real-valued continuous function which is differentiable on  $(0, 1)$  and satisfies  $f(0) = 0$ . Suppose that there exists a constant  $c \in (0, 1)$  such that

$$|f'(x)| \leq c|f(x)| \text{ for every } x \in (0, 1).$$

Show that  $f(x) = 0$  for all  $x \in [0, 1]$ .

71. Suppose  $F : \mathbb{R} \rightarrow \{0, 1\}$  is a function, i.e.  $F$  only takes two values 0 and 1. Suppose  $F$  is non-decreasing, right continuous, and  $\lim_{x \rightarrow \infty} F(x) = 1$ ,  $\lim_{x \rightarrow -\infty} F(x) = 0$ . Prove that there exists some  $x \in \mathbb{R}$  such that  $F(y) = 1$  for all real  $y \geq x$ , and  $F(y) = 0$  for all real  $y < x$ .
72. Consider a container of the shape obtained by revolving a segment (From  $y = 0$  to  $y = 5$ ) of the curve  $x = e^{\frac{y}{5}}$  around the  $y$ -axis. The container is initially empty. Water is poured at a constant rate of  $\pi \text{ cm}^3$  into the container. Let  $h(t)$  be the height of water inside the container at time  $t$ . Find  $h(t)$  as a function of  $t$ .

73. Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be differentiable and satisfy

$$f'(x) = -3f(x) + 6f(2x)$$

for  $x > 0$ . Assume that  $|f(x)| \leq e^{-\sqrt{x}}$  for  $x \geq 0$ . For  $n \in \mathbb{N}$ , define

$$\mu_n = \int_0^\infty x^n f(x) dx.$$

a. Express  $\mu_n$  in terms of  $\mu_0$ . b. Prove that the sequence  $\frac{3^n \mu_n}{n!}$  always converges, and the limit is 0 only if  $\mu_0$ .

74. Suppose  $f : \mathbb{R} \rightarrow [0, \infty)$ . For  $\epsilon > 0$ , define  $f_\epsilon : \mathbb{R} \rightarrow [0, \infty)$  by

$$f_\epsilon(x) = n\epsilon \text{ when } n\epsilon \leq f(x) < (n+1)\epsilon$$

Prove that  $f_\epsilon(x) \rightarrow f(x)$  for all  $x \in \mathbb{R}$  as  $\epsilon \rightarrow 0$ .

75. Suppose  $f_1, f_2, \dots, f_n$  are convex function from  $\mathbb{R}$  to  $\mathbb{R}$ . Prove that  $M(x) = \max\{f_1(x), \dots, f_n(x)\}$  is also convex. [A function is called convex if for all  $x, y \in \mathbb{R}$  and  $\alpha \in [0, 1]$ ,  $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$ ]. Can you say anything about the minimum function? 9 + 1

76. Assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous, one-one function. If there exists a positive integer  $n$  such that  $f^n(x) = x$ , for every  $x \in \mathbb{R}$ , then prove that either  $f(x) = x$  or  $f^2(x) = x$ . (Note that  $f^n(x) = f(f^{n-1}(x))$ .)

77. Suppose  $f : \mathbb{R} \rightarrow [0, \infty)$ . Define the following functions for  $n \in \mathbb{N}$

$$s_n(x) = \begin{cases} n & \text{if } f(x) \geq n \\ 2^{-n}i & \text{if } 2^{-n}i \leq f(x) < 2^{-n}(i+1), i = 0, 1, 2, \dots, n \cdot 2^n - 1 \end{cases}$$

Prove that  $s_1 \leq s_2 \leq s_3 \leq \dots$ . Also, prove that  $\forall x, s_n(x) \rightarrow f(x)$  as  $n \rightarrow \infty$ .

78. Define  $\{x_n\}_{n \in \mathbb{N}}$  as  $x_0 = 1$ , and,  $x_{n+1} = \ln(e^{x_n} - x_n)$ . Prove that  $\sum_{n=0}^\infty x_n$  converges, hence find the value.

79. Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable function such that  $0 \leq f'(x) \leq 1$  and  $f(a) = 0$ . Show that

$$3 \left( \int_a^b f(x)^2 dx \right)^3 \geq \int_a^b f(x)^8 dx.$$

80. Let  $I \subseteq \mathbb{R}$  be an interval and  $f : I \rightarrow \mathbb{R}$  be a differentiable function. Let

$$J = \left\{ \frac{f(b) - f(a)}{b - a} : a, b \in I, a < b \right\}$$

Show that

a)  $J$  is an interval.

b)  $J \subseteq f'(I)$  and  $f'(I) - J$  contains at most two elements.

81. Let  $f$  be a continuous function on  $[0, 1]$  such that for every  $x \in [0, 1]$ , we have  $\int_x^1 f(t) dt \geq \frac{1-x^2}{2}$ .

Show that  $\int_0^1 f(t)^2 dt \geq \frac{1}{3}$ .

82. Let  $(x_n)_{n \in \mathbb{N}}$  be the sequence defined as  $x_n = \sin(2\pi n!e)$  for all  $n \in \mathbb{N}$ . Compute  $\lim_{n \rightarrow \infty} x_n$ .

83. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function which is differentiable at 0. Define another function  $g : \mathbb{R} \rightarrow \mathbb{R}$  as follows:

$$g(x) = \begin{cases} f(x) \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Suppose that  $g$  is also differentiable at 0. Prove that

$$g'(0) = f'(0) = f(0) = g(0) = 0.$$

84. Determine all the pairs of positive real numbers  $(a, b)$  with  $a < b$  such that the following series

$$\sum_{k=1}^{\infty} \int_a^b \{x\}^k dx = \int_a^b \{x\} dx + \int_a^b \{x\}^2 dx + \int_a^b \{x\}^3 dx + \cdots$$

is convergent and determine its value in function of  $a$  and  $b$ .  $\{x\} = x - \lfloor x \rfloor$  denotes the fractional part of  $x$ . Assume you can swap the integral and summation.

85. Determine all continuous functions  $f : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$  that satisfy

$$f(x) = (x+1)f(x^2),$$

for all  $x \in \mathbb{R} \setminus \{-1, 1\}$ .

86. Let  $f : [0, 1] \rightarrow (0, \infty)$  be a continuous function satisfying  $\int_0^1 f(t) dt = \frac{1}{3}$ . Show that there exists  $c \in (0, 1)$  such that  $\int_0^c f(t) dt = c - \frac{1}{2}$ .

87. Determine all ordered pairs of real numbers  $(a, b)$  such that the line  $y = ax + b$  intersects the curve  $y = \ln(1 + x^2)$  in exactly one point.

88. Let  $f : (a, b) \rightarrow \mathbb{R}$  is continuously differentiable,  $\lim_{x \rightarrow a^+} f(x) = \infty$ ,  $\lim_{x \rightarrow b^-} f(x) = -\infty$  and  $f'(x) + f^2(x) \geq -1$  for  $x \in (a, b)$ . Prove that  $b - a \geq \pi$  and give an example where  $b - a = \pi$ .

89. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function whose second derivative is continuous. Suppose that  $f$  and  $f''$  are bounded. Show that  $f'$  is also bounded.

90. Consider the sequence  $(a_n)_{n \geq 1}$  defined by  $a_1 = 1/2$  and  $2n \cdot a_{n+1} = (n+1)a_n$ . Determine the general formula for  $a_n$ . Let  $b_n = a_1 + a_2 + \cdots + a_n$ . Prove that  $\{b_n\} - \{b_{n+1}\} \neq \{b_{n+1}\} - \{b_{n+2}\}$ .

91. If  $f : [0, \infty] \rightarrow \mathbb{N}$  is a right continuous function having left limits. Show that  $\forall t \in [0, \infty)$ , the set  $\{y : f(x) = y, x \in [0, t]\}$  has a finite number of points, i.e., the range for each closed and bounded interval starting from 0 has a finite number of points. [Hint: You can use the fact that each bounded sequence has a convergent subsequence.]