Inequality needed for I.S.I. and C.M.I.

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- **Cauchy-schwarz:**

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2, a_i, b_i \in \mathbb{R}\ \forall i = 1(1)n$$

Equality occurs when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

- **Titu's Lemma:**

$$\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} b_i}, a_i \in \mathbb{R}, b_i > 0\ \forall i = 1(1)n.$$

- **Jensen's Inequality:**

$f: [a, b] \to \mathbb{R}$. $x_1, x_2, \ldots, x_n \in [a, b]$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0,1]$ such that $\sum \alpha_i = 1$. $M = \alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n), N = f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n)$.
If f is convex, then $M \geq N$. If f is concave, $M \leq N$.

- **AM-GM-HM:**

$$AM = \frac{a_1 + a_2 + \cdots + a_n}{n}, GM = \sqrt[n]{a_1 a_2 \ldots a_n},$$

$$HM = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Then $AM \geq GM \geq HM, for\ a_i > 0\ \forall i = 1(1)n$.

Note that: Weighted AM-GM-HM inequality is same as above, if there are w_i many $a_i's$, then in spite of writing

them separately, we take care of the frequencies of a certain value and according to that, the inequality. Up to now, weights are integers. Look at problem 10.

- **Nesbitt's Inequality:**

$$a, b, c > 0. Then \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

And, equality holds iff $a = b = c$.

- **Rearrangement Inequality:**

$$a_1 \geq a_2 \geq \cdots \geq a_n > 0, b_1 \geq b_2 \geq \cdots \geq b_n > 0$$
$$M = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$
$$m = a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1,$$

Let $(c_1, c_2, \dots, c_n)$ be the permutation of $(b_1, b_2, \dots, b_n)$
Then, $M \geq a_1 c_1 + a_2 c_2 + \cdots + a_n c_n \geq m$.

- **Chebyshev Inequality:**

Let $a_1, a_2, \dots, a_n > 0$ & $b_1, b_2, \dots, b_n > 0$.

$$T_1 = \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n}$$
$$T_2 = \frac{(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)}{n^2}$$

Then $T_1 \geq T_2$ if a_i & $b_i's$ are similarly sorted. And, if $a_i, b_i's$ are oppositely sorted, $T_1 \leq T_2$.

- **RMS-AM inequality:**

$$\sqrt{\frac{\sum_{i=1}^{n} a_i^2}{n}} \geq \frac{\sum_{i=1}^{n} a_i}{n}.$$

Some problems:

1)Derive titu's lemma from Cauchy-Schwarz inequality.

2)Prove AM-GM-HM inequality from Jensen's inequality.

3)$a, b, c, d > 0, abcd = 1$. Show that $(a+1)(b+1)(c+1)(d+1) \geq 16$.

4)$a, b, c > 0, ab + bc + ca = 3$. Show that

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq 1.$$

5)$a, b, c > 0$. Prove that $\sum \sqrt{\frac{a}{b+c}} > 2$.

6)$x_1, x_2, \dots, x_n > 0, n \geq 2$. Prove that

$$\frac{1+x_1^2}{1+x_1x_2} + \frac{1+x_2^2}{1+x_2x_3} + \cdots + \frac{1+x_n^2}{1+x_nx_1} \geq n.$$

7)$a, b, c \geq 0, and\ a + b + c \geq abc$. Prove that

$$a^2 + b^2 + c^2 \geq abc$$

8)Which of the following is greater between

$\left(1+\frac{1}{2021}\right)^{2021}$ and $\left(1+\frac{1}{2022}\right)^{2022}$

9)Let $a_1, a_2, \dots, a_n$ be positive reals and $w_1, w_2, \dots, w_n$ be positive rational numbers such that $w_1 + w_2 + \cdots + w_n = 1$. Then,

$$\sum_{i=1}^{n} w_i a_i \geq \prod_{i=1}^{n} a_i^{w_i}.$$

Compare it to weighted AM-GM inequality.

10)Prove that, in weighted AM-GM-HM inequality, the weights may be rational also, i.e. prove the same inequality when weights are positive rational. Hence, conclude that the weights may be rational also as mentioned integer before.

11)$a_1, a_2, \dots \dots, a_n > 0, a_1 + a_2 + \cdots + a_n = 1$. Prove that $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n \leq \frac{1}{4}$.

12)$a_1, a_2, \dots \dots, a_n > 0$. Prove that

$\frac{a_2}{(a_1+a_2)^2} + \frac{a_3}{(a_1+a_2+a_3)^2} + \cdots + \frac{a_n}{(a_1+a_2+\cdots+a_n)^2} \leq \frac{1}{a_1}$.

13)Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two permutations of the numbers 1,2,….,n. Show that

$$\sum_{i=1}^{n} i(n+1-i) \leq \sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} i^2$$

14)If $a, b, c \in (0,1)$ satisfy $a + b + c = 2$. Prove that $\frac{abc}{(1-a)(1-b)(1-c)} \geq 8$.

15)Let $x_1, x_2, \dots, x_n$ be positive real numbers with $x_1 + x_2 + \cdots + x_n = 1$. Then show that

$$\sum_{i=1}^{n} \frac{x_i}{2 - x_i} \geq \frac{n}{2n - 1}.$$

16)For any positive integer n, show that

$$\frac{1}{2} . \frac{3}{4} . \frac{5}{6} \ldots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$

17)For any integer n greater than 1, show that

$$2^n < \binom{2n}{n} < \frac{2^n}{\prod_{i=0}^{(n-1)} \left(1 - \frac{i}{n}\right)}$$

18)For any $k \in \mathbb{Z}^+$, prove that

$2\left(\sqrt{k+1} - \sqrt{k}\right) < \frac{1}{\sqrt{k}} < 2\left(\sqrt{k} - \sqrt{k-1}\right).$

19)Let a, b, c be real numbers greater than 1. Let S denote the sum $S = \log_a bc + \log_b ca + \log_c ab$. Find the smallest possible value of S.