# Polynomials (Problem Set)

Srijan Chattopadhyay

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(*) might require calculus knowledge. (*) are past ISI problems

1. Find all positive integers $a, b$ such that each of the equations

$$x^2 - ax + b = 0 \quad \text{and} \quad x^2 - bx + a = 0$$

   has distinct positive integral roots.

2. $f(x)$ is a degree 4 polynomial satisfying $f(n) = \frac{1}{n}$ for $n = 1, 2, 3, 4, 5$. If $f(0) = \frac{a}{b}$, (where $a$ and $b$ are co-prime positive integers), then what is $a + b$?

3. Find the number of real solutions of the equation:

$$(x-1)(x-3)(x-4)\ldots(x-2025) = (x-2)(x-4)(x-6)\ldots(x-2024)$$

4. Let $a, b$ be the roots of the equation

$$x^2 - 10cx - 11d = 0$$

   and those of

$$x^2 - 10ax - 11b = 0$$

   are $c, d$. Then what is $a + b + c + d$? $\quad (a \neq b \neq c \neq d)$

5. Let $x_1, x_2, \ldots, x_n$ be complex numbers satisfying the equations

$$x_1 + x_2 + \cdots + x_n = n$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 = n$$

$$x_1^3 + x_2^3 + \cdots + x_n^3 = n$$

$$\vdots$$

$$x_1^n + x_2^n + \cdots + x_n^n = n$$

   Then, prove that $x_i = 1 \quad \forall i = 1, 2, \ldots, n$.

6. (*) Let $P(x), Q(x)$ be distinct polynomials with real coefficients such that the sum of the coefficients of each of the polynomials is $s$. If

$$P(x)^3 - Q(x)^3 = P(x^3) - Q(x^3),$$

   then prove that

   - $P(x) - Q(x) = (x-1)^a r(x)$ for some integer $a \geq 1$ and a polynomial $r(x)$ with $r(1) \neq 0$.
   - $s^2 = 3^{a-1}$, where $a$ is as given in the previous.

7. $x_1^2 + px_1 + q = x_2, x_2^2 + px_2 + q = x_3, x_3^2 + px_3 + q = x_1$. Let $p, q$ be real numbers with $\alpha < \beta$ be the roots of the equation $x^2 + (p-1)x + q = 0$. What is the maximum number of solutions of the system of the equations above where $x_1, x_2, x_3 \in [\alpha, \beta]$ is?

8. Consider all the numbers of the form $1 \pm \sqrt{2} \pm \sqrt{3} \pm \cdots \pm \sqrt{2024}$. Prove that when all of the numbers are multiplied, the product will belong to $\mathbb{Z}$.

9. $w, x, y, z \in \mathbb{N}, w^2 + x^2 + y^2 + z^2 = wxy + xyz + wxz + wyz$. Prove that there exists a solution $(w_0, x_0, y_0, z_0)$ such that each of them is greater than $2025^{2025}$. [Hint: Note that $(1, 1, 1, 1)$ is a solution. Now, suppose $(w^*, x^*, y^*, z^*)$ is a general solution. WLOG $x^*$ be the minimum of them. So, fixing $w^*, y^*, z^*$, get another x. Then can you proceed similarly?]

10. $a, b, c \in \mathbb{R}$ such that $(a + c)(a + b + c) < 0$. Prove that $\left(\frac{b-c}{2}\right)^2 \geq a(a + b + c)$.

11. Let $P(x)$ be a polynomial such that $(x + 1)P(x - 1) = (x - 1)P(x)$ for all $x \in \mathbb{R}$. Determine the maximum possible degree of $P(x)$.

12. Show that the quadratic equation $x^2 + 7x - 14(q^2 + 1) = 0, q \in \mathbb{Z}$ has no integer root.

13. (*)(*) Consider the equation $x^5 + x = 10$. Show that
    (a) The equation has only one real root.
    (b) The root lies between 1 and 2.
    (c) This root must be irrational.

14. $f(x)$ i sa cubic polynomial $x^3 + ax^2 + bx + c$ such that $f(x) = 0$ has three distinct integral roots and $f(g(x)) = 0$ doesn't have any real roots, where $g(x) = x^2 + 2x - 5$. Then find minimum value of $a + b + c$.

15. Let $f(x)$ be a polynomial with integer coefficients such that $f(2) = f(0) = f(1) = f(5) = n$ and $f(-2) = f(-1) = f(-5) = -n$ for some positive integer $n$. Find the smallest possible value of $n$.

16. Let $p(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - 4x^{2n-3} + \cdots - 2nx + (2n + 1)$ Show that the polynomial $p(x)$ has no real root.

17. Show that the polynomial equation with real coefficients $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_3x^3 + x^2 + x + 1 = 0$ cannot have all real roots.

18. Let $a_1, \ldots, a_n \geq 0$, not all zero. Show that the equation
$$x^n - a_1x^{n-1} - a_2x^{n-2} - \cdots - a_n = 0$$
has a unique positive real root. Moreover, if $r$ be that root, then show that $r^b \geq a^a$ where $a = \sum_{j=1}^n a_j$ and $b = \sum_{j=1}^n ja_j$.

19. Let $P(x) = x^{100} + 20x^{99} + 198x^{98} + a_{97}x^{97} + \ldots + a_1x + 1$ be a polynomial where the $a_i$ $(1 \leq i \leq 97)$ are real numbers. Prove that the equation $P(x) = 0$ has at least one nonreal root.

20. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients such that $f(1) = -1, f(4) = 2$ and $f(8) = 34$. Suppose $n \in \mathbb{Z}$ is an integer such that $f(n) = n^2 - 4n - 18$. Determine all possible values for $n$.

21. Let $a_1, \ldots, a_n \geq 0$, not all zero. Show that the equation
$$x^n - a_1x^{n-1} - a_2x^{n-2} - \cdots - a_n = 0$$
has a unique positive real root. Moreover, if $r$ be that root, then show that $r^b \geq a^a$ where $a = \sum_{j=1}^n a_j$ and $b = \sum_{j=1}^n ja_j$.

22. (*) $P(x) \in \mathbb{Z}[x], P(1) = 7$, and $P(n)$ is prime for all $n \in \mathbb{N}$. Find $P(2025)$.

23. $f, g \in \mathbb{R}[x]$, and $f$ is non zero polynomial. Also, we have $f(x^2 + x + 1) = f(x).g(x)$ for all $x \in \mathbb{R}$. Prove that $deg(g)$ must be even.

24. Suppose $a, b, c$ are distinct integers. $P(x) \in \mathbb{Z}[x]$. Also, we have $P(a) = P(b) = P(c) = -1$. Find all integer roots of $P$.

25. $0 < a \le b \le c \in \mathbb{R}$. Also, $a \le x \le y \le z \le c$, where $x, y, z \in \mathbb{R}$. It is given that $a+b+c = x+y+z$, and $xyz = abc$. Prove that $a = x, b = y, c = z$.

26. $a_1, a_2, \cdots, a_{2025}$ are distinct reals. $P(x) \in \mathbb{R}[x]$. Degree of $P$ is 2024. Given that $P(1) = 2026$, and $|P(a_i) - P(a_j)| = |a_i - a_j|$ for all $i, j \in \{1, \cdots, 2025\}$. Find all such polynomials $P$.

27. $P(x) \in \mathbb{Z}[x]$. $a, b, c$ distinct integers such that $P(a) = b, P(b) = c, P(c) = a$. Prove that such polynomials don't exist.

28. $a \in \mathbb{Z}, P(x) \in \mathbb{Z}[x]$ such that $P(P(P(P(a)))) = a$. Prove that $P(P(a)) = a$.

29. $f$ is a polynomial such that $f(x) \in \mathbb{Z}[x]$ such that $f(0), f(1)$ both are odd. Find all integer roots of $f$.

30. $f$ is a polynomial such that $f(x) \in \mathbb{R}[x]$. $f(1) = 3$ and $f(x+1) = f(x) + 3x^2 + 3x + 1$. Find $f(\frac{1}{2})$.

31. $f$ is a polynomial such that $f(x) \in \mathbb{R}[x]$. Degree of $f$ is n. Also, $f(i) = \frac{i}{i+1}$ for $i = 0, 1, \cdots, n$. Find $f(n+1)$.

32. $P(x) = x^n + x^{n-1} + x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_1 x + a_0$ for $n \ge 3$. Can all roots of $P$ be real?

33. $f(x) = x^n - nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_1 x + a_0$. It is given that all roots of $f$ are real. Find $f$.

34. If the sum of the real roots $x$ to each of the equations

$$2^{2x} - 2^{x+1} + 1 - \frac{1}{k^2} = 0$$

for $k = 2, 3, \ldots, 2023$ is $N$, what is $2^N$?

35. Let $x, y, z$ be nonzero numbers, not necessarily real, such that

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 24yz$$

and

$$\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = 3.$$

Compute $\dfrac{x^2}{yz}$.

36. Suppose that $p(x), q(x)$ are monic polynomials with nonnegative integer coefficients such that

$$\frac{1}{5x} \ge \frac{1}{q(x)} - \frac{1}{p(x)} \ge \frac{1}{3x^2}$$

for all integers $x \ge 2$. Compute the minimum possible value of $p(1) \cdot q(1)$.

37. Let the roots of

$$x^{2022} - 7x^{2021} + 8x^2 + 4x + 2$$

be $r_1, r_2, \ldots, r_{2022}$, the roots of

$$x^{2022} - 8x^{2021} + 27x^2 + 9x + 3$$

be $s_1, s_2, \ldots, s_{2022}$, and the roots of

$$x^{2022} - 9x^{2021} + 64x^2 + 16x + 4$$

be $t_1, t_2, \ldots, t_{2022}$. Compute the value of

$$\sum_{1 \le i,j \le 2022} r_i s_j + \sum_{1 \le i,j \le 2022} s_i t_j + \sum_{1 \le i,j \le 2022} t_i r_j.$$

38. Let $f^1(x) = x^3 - 3x$. Let $f^n(x) = f(f^{n-1}(x))$. Let $\mathcal{R}$ be the set of roots of

$$\frac{f^{2022}(x)}{x}.$$

If

$$\sum_{r \in \mathcal{R}} \frac{1}{r^2} = \frac{a^b - c}{d}$$

for positive integers $a, b, c, d$, where $b$ is as large as possible and $c$ and $d$ are relatively prime, find $a + b + c + d$.

39. Let $x, y, z$ be positive real numbers with $1 < x < y < z$ such that

$$\log_x y + \log_y z + \log_z x = 8, \quad \text{and}$$

$$\log_x z + \log_z y + \log_y x = \frac{25}{2}.$$

The value of $\log_y z$ can then be written as $\dfrac{p + \sqrt{q}}{r}$ for positive integers $p, q$, and $r$ such that $q$ is not divisible by the square of any prime. Compute $p + q + r$.

40. Find the sum of all possible values of $a$ such that there exists a non-zero complex number $z$ such that the four roots, labeled $r_1$ through $r_4$, of the polynomial

$$x^4 - 6ax^3 + (8a^2 + 5a)x^2 - 12a^2x + 4a^2$$

satisfy $|\Re(r_i)| = |r_i - z|$ for $1 \leq i \leq 4$. Note, for a complex number $x$, $\Re(x)$ denotes the real component of $x$.

41. Suppose that the polynomial $x^2 + ax + b$ has the property such that if $s$ is a root, then $s^2 - 6$ is a root. What is the largest possible value of $a + b$?

42. Suppose $f(x)$ is a monic quadratic polynomial such that there exists an increasing arithmetic sequence $z_1 < z_2 < z_3 < z_4$ where $|f(z_1)| = |f(z_2)| = |f(z_3)| = |f(z_4)| = 2020$. Compute the absolute difference of the two roots of $f(z)$.

43. (*)(*) Let $a, b, c$ be three real numbers which are roots of a cubic polynomial, and satisfy $a+b+c = 6$ and $ab + bc + ca = 9$. Suppose $a < b < c$. Show that

$$0 < a < 1 < b < 3 < c < 4.$$

44. (*) Let $a_0, a_1, \ldots, a_{19} \in \mathbb{R}$ and

$$P(x) = x^{20} + \sum_{i=0}^{19} a_i x^i, x \in \mathbb{R}.$$

If $P(x) = P(-x)$ for all $x \in \mathbb{R}$, and

$$P(k) = k^2,$$

for $k = 0, 1, 2, \ldots, 9$. Find $P(x)$.

45. (*) Let $c$ be a fixed real number. Show that a root of the equation

$$x(x+1)(x+2)\cdots(x+2009) = c$$

can have multiplicity at most 2. Determine the number of values of $c$ for which the equation has a root of multiplicity 2.

46. (*) Let $P(X)$ be a polynomial with integer coefficients of degree $d > 0$.

$(a)$ If $\alpha$ and $\beta$ are two integers such that $P(\alpha) = 1$ and $P(\beta) = -1$ , then prove that $|\beta - \alpha|$ divides 2.

$(b)$ Prove that the number of distinct integer roots of $P^2(x) - 1$ is atmost $d + 2$.

47. (*) We are given $a, b, c \in \mathbb{R}$ and a polynomial $f(x) = x^3 + ax^2 + bx + c$ such that all roots (real or complex) of $f(x)$ have same absolute value. Show that $a = 0$ iff $b = 0$.

48. (*) Let $f(x)$ be a polynomial with integer co-efficients. Assume that 3 divides the value $f(n)$ for each integer $n$. Prove that when $f(x)$ is divided by $x^3 - x$ , the remainder is of the form $3r(x)$ where $r(x)$ is a polynomial with integer coefficients.

49. (*) Let $f(x) = ax^2 + bx + c$ where $a, b, c$ are real numbers. Suppose $f(-1), f(0), f(1) \in [-1, 1]$. Prove that $|f(x)| \leq \frac{3}{2}$ for all $x \in [-1, 1]$.

50. (*) Suppose that $P(x)$ is a polynomial with real coefficients, such that for some positive real numbers $c$ and $d$, and for all natural numbers $n$, we have $c|n|^3 \leq |P(n)| \leq d|n|^3$.

    Prove that $P(x)$ has a real zero.

51. (*) If a polynomial $P$ with integer coefficients has three distinct integer zeroes , then show that $P(n) \neq 1$ for any integer $n$.

52. (*) Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial such that $P(X) = X$ has no real solution. Prove that $P(P(X)) = X$ has no real solution.

53. (*) Let $a, b, c$ be nonzero real numbers such that $a + b + c \neq 0$. Assume that

    $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}$$

    Show that for any odd integer $k$,

    $$\frac{1}{a^k} + \frac{1}{b^k} + \frac{1}{c^k} = \frac{1}{a^k + b^k + c^k}.$$

54. (*) Let $k, n$ and $r$ be positive integers.

    (a) Let $Q(x) = x^k + a_1 x^{k+1} + \cdots + a_n x^{k+n}$ be a polynomial with real coefficients. Show that the function $\frac{Q(x)}{x^k}$ is strictly positive for all real $x$ satisfying

    $$0 < |x| < \frac{1}{1 + \sum\limits_{i=1}^{n} |a_i|}$$

    (b) Let $P(x) = b_0 + b_1 x + \cdots + b_r x^r$ be a non zero polynomial with real coefficients. Let $m$ be the smallest number such that $b_m \neq 0$. Prove that the graph of $y = P(x)$ cuts the $x$-axis at the origin (i.e., $P$ changes signs at $x = 0$) if and only if $m$ is an odd integer.

55. If $\alpha$ is a root of the polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with real coefficients, $a_n \neq 0$, then prove that

    $$|\alpha| \leq 1 + \max_{0 \leq k \leq n-1} \left| \frac{a_k}{a_n} \right|.$$

56. Let $n$ be an even positive integer. Let $p$ be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1} x^{2n-1} + \cdots + a_1 x + a_0$ for some real coefficients $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers $k$ such that $1 \leq |k| \leq n$. Find all other real numbers $x$ for which $p(1/x) = x^2$

57. Determine all polynomials $P(x)$ such that $P(x^2 + 1) = (P(x))^2 + 1$ and $P(0) = 0$.

58. $P(x)$ is a polynomial in $x$ with non-negative integer coefficients. If $P(1) = 5$ and $P(P(1)) = 177$, what is the sum of all possible values of $P(10)$?

59. (*) Consider the polynomial $ax^3 + bx^2 + cx + d$ where $a, b, c, d$ are integers such that $ad$ is odd and $bc$ is even.Prove that not all of its roots are rational.

60. (*) If $P(x) = x^n + a_1 x^{n-1} + ... + a_{n-1}$ be a polynomial with real coefficients and $a_1^2 < a_2$ then prove that not all roots of $P(x)$ are real.

61. (*) Let $p(x) = x^7 + x^6 + b_5 x^5 + \cdots + b_0$ and $q(x) = x^5 + c_4 x^4 + \cdots + c_0$ . If $p(i) = q(i)$ for $i = 1, 2, 3, \cdots, 6$ . Show that there exists a negative integer r such that $p(r) = q(r)$ .