Trigonometry needed for I.S.I. and C.M.I. entrance

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Theory of trigonometry needed is as you have completed in board syllabus. So, lets focus on interesting and important problems.

Problems:

1) Two train lines intersect each other at a junction at an acute angle θ. A train is passing along one of the two lines. When the front of the train is at the junction, the train subtends an angle α at a station on the other line. It subtends an angle β(<α) at the same station, when its rear is at the junction, show that $\tan\theta = \frac{2\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$

2) Consider a regular heptagon (polygon of 7 equal sides and equal angles) ABCDEFG.

A) Prove $\frac{1}{\sin\frac{\pi}{7}} = \frac{1}{\sin\frac{2\pi}{7}} + \frac{1}{\sin\frac{3\pi}{7}}$

B) Using A) or otherwise, show that $\frac{1}{AG} = \frac{1}{AF} + \frac{1}{AE}$.

3) Show that the triangle whose angles satisfy the equality

$\frac{\sin^2 A+\sin^2 B+\sin^2 C}{\cos^2 A+\cos^2 B+\cos^2 C} = 2$ is right-angled.

4) Let $a \geq 0$ be a constant such that $\sin\sqrt{x+a} = \sin\sqrt{x}$ for all $x \geq 0$. What can you say about a? Justify your answer.

5) Let X, Y, Z be the angles of a triangle.

i) Prove that $\tan\frac{X}{2}\tan\frac{Y}{2} + \tan\frac{Y}{2}\tan\frac{Z}{2} + \tan\frac{Z}{2}\tan\frac{X}{2} = 1.$

ii) Using i) or otherwise prove that $\tan\frac{X}{2}\tan\frac{Y}{2}\tan\frac{Z}{2} \leq \frac{1}{3\sqrt{3}}$.

6) For $x \geq 0$, define $f(x) = \frac{1}{x+2\cos x}$. Determine the set

$$\{y \in \mathbb{R} : y = f(x), x \geq 0\}$$

7) Let a, b, c be the sides of a triangle and A, B, C be the angles opposite to these sides respectively. If

$$\sin(A - B) = \frac{a}{a+b}\sin A\cos B - \frac{b}{a+b}\cos A\sin B$$

Then prove that the triangle is isosceles.

8) Let the sequence $\{a_n\}_{n\geq 1}$ be defined by $a_n = \tan(n\theta)$ where $\tan\theta = 2$. Show that for all n, a_n is a rational number which can be written with an odd denominator.

9) Find all pairs (x, y) with x, y real, satisfying the equations:

$$\mathrm{Sin}\frac{x+y}{2} = 0, |x| + |y| = 1.$$

10) For all natural numbers n, let

$$A_n = \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}\ [n\ many\ radicals]$$

a) Show that for $n \geq 2, A_n = 2\sin\frac{\pi}{2^{n+1}}$

b) Evaluate the limit $\lim_{n\to\infty} 2^n A_n$

11) Find all solutions of $\sin^5 x + \cos^3 x = 1$.

12) Show that $-2 \leq \cos\theta\,(\sin\theta + \sqrt{\sin^2\theta + 3}) \leq 2$ for all values of θ.

13)Find the average of the number $n \sin n°$ for n=2,4,6,..,180.

14)Find the value of

$$\prod_{k=1}^{n}\left(1 + 2\cos 2\pi . \frac{3^k}{3^n + 1}\right)$$

15)Prove that

$$\left(\frac{1}{2} + \cos\frac{\pi}{20}\right)\left(\frac{1}{2} + \cos\frac{3\pi}{20}\right)\left(\frac{1}{2} + \cos\frac{9\pi}{20}\right)\left(\frac{1}{2} + \cos\frac{27\pi}{20}\right) = \frac{1}{16}$$

16)Evaluate : $10\sum_{1 \leq k < s \leq 1007} \cos\frac{2\pi k}{2015} \cos\frac{2\pi s}{2015}$

17)Prove that, $\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2+\sqrt{2}}}, \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}$ ... ... up to infinite factors= $\frac{\pi}{2}$

18)Evaluate:

$$\sum_{n=1}^{\infty} 4^n \sin^4 \frac{\pi}{2^n}$$

19)Given $x_1, x_2, \dots \dots, x_{2016}$ are real numbers such that $x_i \in [-1,1]\ \forall i$. If $\sum_{i=0}^{2016} x_i^3 = 0$, then find the greatest value of $\sum_{i=0}^{2016} x_i$

20)If $P_n = \prod_{1 \leq k \leq n, \gcd(k,n)=1} \sin\frac{k\pi}{n}, find\ P_{100}$.

21) Let $\theta_1, \theta_2, \theta_3$ be three distinct solution to the equation $\tan\theta = \frac{17}{100}$ *such that*

$$\theta_1, \theta_2, \theta_3 \in (0{,}3\pi). \textit{Find value of } \sum_{1\ i<j}^{3} \tan\left(\frac{\theta_i}{3}\right)\tan\left(\frac{\theta_j}{3}\right)$$

22)The equation $ax^4 - bx^3 - cx^2 + dx + 1 = 0$ has a root of $\cos\frac{2\pi}{15}$ for positive integer a, b, c, d. Find $a + b + c + d$.

23)Show that there is no polynomial p(x) for which $\cos\theta = P(\sin\theta)$ for all angles θ in some non-empty interval.

24) Recall the function arctan(x), also denoted as $\tan^{-1} x$

Complete the sentence:

$\tan^{-1} 20202019 + \tan^{-1} 20202021\ 2(-) \tan^{-1} 20202019$ because in the relevant region, the graph of y = arctan(x). Fill in the first blank with one of the following: is less than / is equal to / is greater than. Fill in the second blank with a single correct reason consisting of one of the following phrases: is bounded / is continuous / has positive first derivative / has negative first derivative / has positive second derivative / has negative second derivative / has an inflection point.

25)Three positive real numbers x, y *and* z satisfy $x^2 + y^2 = 3^2, y^2 + yz + z^2 = 4^2, x^2 + \sqrt{3}xz + z^2 = 5^2$. Find the value of $2xy + xz + \sqrt{3}yz$.

26)Recall that $\sin^{-1} t$ is a function with domain $[-1{,}1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Consider the function $f(x) = \sin^{-1}\sin(x)$. Find where f is well defined, continuous and differentiable.

27)Find the number of real solutions of $x = 99 \sin \pi x$.

28) If $x = cos1°cos2°cos3° \dots \dots .. cos89°\ and\ \ y = cos2°cos6°cos10° \dots \dots . cos86°$ Then what is the integer nearest to $\frac{2}{7}\log_2(\frac{y}{x})$?

29) $a_1, a_2, \dots \dots a_n$ are real numbers either +1 or -1. Prove that

$$2\sin\left[\left(a_1 + \frac{a_1a_2}{2} + \frac{a_1a_2a_3}{2^2} + \cdots \frac{a_1a_2a_3\dots a_n}{2^{n-1}}\right)\frac{\pi}{4}\right] = a_1\sqrt{2 + a_2\sqrt{2 + a_3\sqrt{\dots \dots . + a_n\sqrt{2}}}}.$$