Assignment-1

Date: 10.08.22

1) f(x) is a degree 4 polynomial satisfy $f(n) = \frac{1}{n}$ for n=1,2,3,4,5. If $f(0) = \frac{a}{b}$,(a and b are co-prime positive integers), then a+b =?
2) Find the number of real solutions of the equation:
$$(x-1)(x-3)\ldots(x-2021) = (x-2)(x-4)\ldots(x-2022)$$
3) What is the minimum value of p(2) if the following conditions are followed?
*P(x) is a polynomial of degree 17.
*all roots of P(x) are real.
*all coefficients are positive
*the coefficient of x^{17} is 1
*the product of the roots of p(x) is -1.

4) Let $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n = m$ be positive integers. Denote by b_k the number of these a_i for which $a_i \geq k$. Find $\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} b_i$.

5) Which of the polynomials $(1 + x^2 - x^3)^{1000}$ or $(1 - x^2 + x^3)^{1000}$ has the greater coefficient of x^{20} after expansion and collecting the terms?

6) Quadratic polynomials $P(x)$ and $Q(x)$ have leading coefficients of 2 and -2, respectively. The graphs of both polynomials pass through the two points $(16, 54)$ and $(20, 53)$. Find $P(0) + Q(0)$.

7) Given the polynomial

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$$

With real coefficients, and $a_1^2 < a_2$, show that not all roots of $f(x)$ can be real.

Assignment-2

Date: 17.08.22

1)$(x^3 + 3x^2 + 3x + 2)P(x-1) = (x^3 - 3x^2 + 3x - 2)P(x)$. Let $P(x)$ be a polynomial with real co-efficients such that it satisfies the above equation $\forall x \in \mathbb{R}$. If $P(2) = 84$, then $P(10) = ?$

2)A monic polynomial $f(x)$ of degree 4 satisfies $f(1) = 10, f(2) = 20, f(3) = 30$. Determine $f(12) + f(-8) - 19000$.

3) Let P(x) and Q(x) be distinct polynomials with real coefficients such that the sum of the coefficients of each of the polynomials is S. If $P(x)^3 - Q(x)^3 = P(x^3) - Q(x^3)$,then
i)Prove that $P(x) - Q(x) = (x-1)^a R(x)$ for some integer $a \geq 1$ and a polynomial R(x) with R(1) non-zero.
ii)$S^2 = 3^{a-1}$, where a is as given in i).

4) Suppose $f : \mathbb{R} \to \mathbb{R}$ such that $f\big(f(x)\big) = \big(f(x)\big)^{2013}$. Show that there are infinitely many functions, of which exactly 4 are polynomials.

Assignment

Combinatorics

1)Let $A = \{1,2,..,n\}$. For a permutation $P = \big(P(1), P(2),..,P(n)\big)$ of the elements of A, let $P(1)$ denote the first element of P. Find the number of all such permutations P so that for all $i, j \in A$:

- If $i < j < P(1)$, then j appears before i in P
- If $P(1) < i < j$, then i appears before j in P.

2)Find the explicit form of the sequence $\{x_n\}$ satisfying

$$x_n = (\alpha + \beta)x_{n-1} - \alpha\beta x_{n-2}$$

Where $\alpha, \beta \in \mathbb{R}$

3)Find the number of integral solutions of the following:

$$x_1 + x_2 + \cdots + x_r = n; x_1 \geq b_1, x_2 \geq b_2, \dots, x_r \geq b_r$$

4) 10 candidates participate for Olympiad, which is organised around a table. There are 5 versions of the test and each candidate will receive exactly one version. No 2 consecutive candidates will get same version. How many ways are there to give the questions?

5) There are N boxes, each containing at most r balls. If the number of boxes containing at least i balls is N_i $\forall i = 1(1)r$. Then find total number of balls contained in these N boxes.

6)Find the number of non-negative solutions of $3x + y + z = 24$.

7) Let n be an odd natural number. Suppose that A is an $n \times n$ symmetric matrix such that each row of A is a permutation of 1,2,……,n. Show that the diagonal elements $(a_{11}, a_{22}, \ldots.., a_{nn})$ must also form a permutation of (1,2,3,…..n).

8) A staircase has n steps. A man climbs either one step or two steps at a time. Prove that the number of ways he can climb up the staircase, starting from the bottom, is $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right], n \geq 1.$

9) If $S(n) = \lim_{x \to 0} \sum_{r=1}^{n} \frac{\binom{n}{r} \sin rx \cos(n-r)x}{x.2^n}$, find S(2022).

Complex Number

Assignment

1) Let complex numbers α and $\frac{1}{\tilde{\alpha}}$ lie on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$ respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha|$=?
2) $x_1, x_2, \ldots \ldots \ldots \ldots x_n$ be complex numbers to satisfy the following set of equations
   $x_1 + x_2 + \cdots \ldots \ldots x_n = n$
   $x_1^2 + x_2^2 + \cdots \ldots \ldots x_n^2 = n$
   $x_1^3 + x_2^3 + \cdots \ldots \ldots x_n^3 = n$
   …..
   ……
   $x_1^n + x_2^n + \cdots \ldots \ldots x_n^n = n$
   Then, prove that $x_i = 1 \; \forall i = 1(1)n$
3) If z_1, z_2, z_3 are non-zero complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$.
   Then prove that z_1, z_2, z_3 lie on a circle passing through the origin.
4) If $|z| \geq 3$, then determine the least value of $\left|z + \frac{1}{z}\right|$
5) If $|z - 2 - 3i| + |z + 2 - 6i| = 4$, then show that the locus of z is null, that is no such z exists.
6) If a, b, c are distinct integers and w(not equal to 1) is a cube root of unity, then find the minimum value of
   $$|a + bw + cw^2| + |a + bw^2 + cw|$$
7) If $a_1, a_2, \ldots, a_n$ are distinct integers then show that $P(x) = (x - a_1)(x - a_2) \ldots (x - a_n) - 1$ is irreducible in $Z[x]$.

Number Theory

Assignment

1) Let the divisors of n be $1 = d_1 < d_2 < \cdots < d_k = n$. Let us define the following set $N_n = \{1,2,..,n\}$ and $S_i = \{x: x \in N_n \; and \; \gcd(x,n) = d_i\} \; \forall i = 1,2,\ldots,k$.

i) Prove that S_i's are mutually disjoint.

ii) Prove that, $|S_i| = \emptyset\left(\frac{n}{d_i}\right)$

iii) Prove that $\bigcup_{i=1}^{k} S_i = N_n$.

iv) Hence, prove that $\emptyset(d_1) + \emptyset(d_2) + \cdots + \emptyset(d_k) = n$.

2) Integers a, b, c satisfy $a + b - c = 1, a^2 + b^2 - c^2 = -1$. Then what is sum of all possible distinct values of $(a^2 + b^2 + c^2)$?

3) Show that the quadratic equation $x^2 + 7x - 14(q^2 + 1) = 0 (q \in \mathbb{Z})$ has no integral root.

4) If n is a natural number, prove that $n - sum\ of\ digits\ of\ n$ is always divisible by 9.

5) $N = 13 \times 17 \times 41 \times 829 \times 56659712633$. It is known that N is a 18 digit number with 9 of the ten digits from 0 to 9 each appearing twice. Find the sum of the digits of N.

6) Let $a_1, a_2, \ldots \ldots . a_n$ be integers. Show that there exist integers k and r such that the sum $a_k + a_{k+1} + \cdots \ldots \ldots . a_{k+r}$ $is\ divisible\ by\ n$

7) $f(x) = \begin{cases} 0 + \beta : \lfloor x \rfloor = 0\ mod\ 3 \\ 1 + \beta : \lfloor x \rfloor = 1\ mod\ 3 \\ 2 + \beta : \lfloor x \rfloor = 2\ mod\ 3 \end{cases}$ where $\lfloor x \rfloor$ denotes the greatest integer function. If $\sum_{n=1}^{\infty} \frac{f(3^n\sqrt{2021})}{3^n} = 0$, then find value of β.

8) Consider a right angled triangle with integer valued sides a<b<c, with a, b, c pairwise co prime. Let d = c – b. Suppose $d|a$. Then
A) Prove that $d \leq 2$.
B) Find all such triangles (i.e. all possible triples a, b, c) with perimeter less than 100

9) Let $n = 2^{31} \times 3^{19}$. How many divisors of n^2 are less than n but do not divide n?

10) Let $a_1, a_2, a_3, \dots . a_n$ be positive integers which satisfies $a_1 < a_2 < a_3 < \cdots < a_n$ and $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots \frac{1}{a_n} = 1$. Find a general method to find a tuple $(a_1, a_2, \dots . a_n)$ satisfying the above property.

11) Let $a = \sqrt[2022]{2022}$ which is greater between 2022 and $a^{a^{\cdot^{\cdot^{a}}}}$, where a appears 2022 times.

12) Consider all non-empty subsets of the set {1,2,…,n}. For every such subset, we find the product of reciprocals of each of its elements. Denote the sum of all these products by S_n. For example, $S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1.2} + \frac{1}{1.3} + \frac{1}{2.3} + \frac{1}{1.2.3}$

A)Show that $S_n = \frac{1}{n} + \left(1 + \frac{1}{n}\right) S_{n-1}$.
B)Prove using (A) that $S_n = n$.
C)Prove not using (A) that $S_n = n$.

13) How many distinct integers are in the sequence
$\left\lfloor \frac{1^2}{2022} \right\rfloor, \left\lfloor \frac{2^2}{2022} \right\rfloor, \left\lfloor \frac{3^2}{2022} \right\rfloor, \dots \dots, \left\lfloor \frac{2022^2}{2022} \right\rfloor$

14) Suppose the set $\{1,2,3, \dots \dots 1998\}$ is partitioned into disjoint pairs $\{a_i, b_i\}$ $(1 \leq i \leq 999)$ in a manner that for each i, $|a_i - b_i|$ equals 1 or

6. Determine with proof, the last digit of the sum $S = |a_1 - b_1| + |a_2 - b_2| + \cdots |a_{100} - b_{100}|$.

15) $\sum_{i=1}^{n}(\left\lfloor\frac{n}{i}\right\rfloor - \left\lfloor\frac{n-1}{i}\right\rfloor) = A, \sum_{i=1}^{n}\left(\left\lfloor\frac{n}{i}\right\rfloor^2 - \left\lfloor\frac{n-1}{i}\right\rfloor^2\right) = B$, where n is a natural number. Prove that number of divisor is A, and sum of divisors of n is $\frac{A+B}{2}$

16) Let p be a prime number bigger than 5. Suppose the decimal expansion of $\frac{1}{p}$ looks like $0.\overline{a_1 a_2 \dots a_r}$ where the line denotes a recurring decimal. Prove that 10^r leaves a remainder of 1 on dividing by p.

17) Prove that every positive rational number can be expressed uniquely as a finite sum of the form

$$a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!},$$

Where a_n are integers such that $0 \leq a_n \leq n - 1\ \forall n > 1$

18) Let $n \geq 2$ be an integer. Let m be the largest integer which is less than or equal to n, and which is a power of 2. Put l_n= least common multiple of 1,2,…,n. Show that $\frac{l_n}{m}$ is odd, and that for every integer $k \leq n, k \neq m, \frac{l_n}{k}$ is even. Hence, prove that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not an integer.