Assignment-1

Date: 02.08.22

1) Suppose a, b are integers and a + b is a real root of $x^2 + ax + b = 0$, what is the maximum possible value of b^2?
2) Prove that there exist 100 consecutive natural numbers, such that exactly 3 of them are primes.
3) A series is formed in the following manner:
A(1)=1; and A(n)=f(m) number of f(m) followed by f(m) number of 0; where m is the number of digits in A(n-1) and f(m) is the remainder when m is divided by 9. Find sum of digits of A(30).
4) Find the last non-zero digit of 2022!
5)

1→	2↓	9→	10	25	….
4↓	3←	8↑	11	24	
5→	6→	7↑	12	23	
16	15	14	13	22	
17	18	19	20	21	

In which row will 2022 appear if the pattern indicated by the arrows above is followed?

6) Rahul and Rohit just became friends with Neha and they want to know when her birthday is. Neha gives them a list of possible dates,

May 15|May 16|June 17|June 18|July 14|July 16|July 19|Aug 14|Aug 15|Aug 17

Neha then tells Rahul and Rohit separately the month and date of her birthday respectively.

Rahul: I don't know when Neha's birthday is, but I know that Rohit doesn't know also.

Rohit: At first, I don't know when Neha's birthday is, but I know now.

Rahul: Then I also know when it is.

So, when is Neha's birthday?

7)Let the divisors of n be $1 = d_1 < d_2 < \cdots < d_k = n$. Let us define the following set $N_n = \{1,2,..,n\}$ and $S_i = \{x: x \in N_n \ and \ \gcd(x,n) = d_i\} \ \forall i = 1,2,\ldots,k$.

i) Prove that S_i's are mutually disjoint.

ii) Prove that, $|S_i| = \emptyset\left(\frac{n}{d_i}\right)$

iii) Prove that $\bigcup_{i=1}^{k} S_i = N_n$.

iv) Hence, prove that $\emptyset(d_1) + \emptyset(d_2) + \cdots + \emptyset(d_k) = n$.

v) Without using i) to iii) try to prove iv) just using prime factorization. [Hint: $\emptyset(mn) = \emptyset(m)\emptyset(n), if \ \gcd(m,n) = 1$]

Assignment-2

Date: 09.08.22

1) $f: \mathbb{N} \to \mathbb{N}$, such that $f(ab) = f(a)f(b)$ if $\gcd(a, b) = 1$.

$$\sum_{d|n} f(d) = n.$$

Prove that $f(n) = \emptyset(n)$ is the only solution, where $\emptyset$ is the euler's totient function.

2) Let p be a prime number bigger than 5. Suppose the decimal expansion of $\frac{1}{p}$ looks like $0.\overline{a_1 a_2 \dots a_r}$ where the line denotes a recurring decimal. Prove that 10^r leaves a remainder of 1 on dividing by p.

3) $(x^3 + 3x^2 + 3x + 2)P(x-1) = (x^3 - 3x^2 + 3x - 2)P(x)$. Let $P(x)$ be a polynomial with real co-efficients such that it satisfies the above equation $\forall x \in \mathbb{R}$. If $P(2) = 84, P(10) = ?$

4) $n \in \mathbb{N}, n$ has k divisors. $d_1 < d_2 < \cdots < d_{k,}$ and $d_5^2 + d_6^2 = 2n + 1$

$i)$ Prove that $d_5 < \sqrt{n}$ and $d_6 > \sqrt{n}$.

$ii)$ Hence, find the number of divisors of n.

$iii)$ Find k and n.

5) You are in a dark room where a table is kept. There are 50 coins placed on a table out of which 10 coins are showing tails and 40 are showing heads. The task is to divide this set of 50 coins into two groups (not necessarily of same size) such that both groups have same number of coins showing the tails.

6)$f(x) \in \mathbb{R}[x]$, $\deg(f) = n$. $f(i) = \frac{i}{i+1}$ $\forall i = 0,1,..,n$. Then find $f(n+1)$.

Assignment-3

Date: 16.08.22

1) Consider the equation $x^5 + x - 10 = 0$. Prove that the equation has only one root and the root of the equation must be irrational.

2) Find last 3 digits of 7^{9999}.

3) $x_1, x_2, \dots \dots \dots \dots x_n$ be complex numbers to satisfy the following set of equations

$$x_1 + x_2 + \cdots \dots \dots x_n = n$$
$$x_1^2 + x_2^2 + \cdots \dots \dots x_n^2 = n$$
$$x_1^3 + x_2^3 + \cdots \dots \dots x_n^3 = n$$

.....

......

$$x_1^n + x_2^n + \cdots \dots \dots x_n^n = n$$

Then, prove that $x_i = 1\ \forall i = 1(1)n$.

4) Let P(x) be the polynomial when $(x+7)^{100}$ is divided by $(x^2 - x - 1)$. Now find the remainder when P(x) is divided by 11.

5) Let P(x) and Q(x) be distinct polynomials with real coefficients such that the sum of the coefficients of each of the polynomials is S. If $P(x)^3 - Q(x)^3 = P(x^3) - Q(x^3)$,then

i)Prove that $P(x) - Q(x) = (x-1)^a R(x)$ for some integer $a \geq 1$ and a polynomial R(x) with R(1) non-zero.

ii)$S^2 = 3^{a-1}$, where a is as given in i).

Assignment-4

Date: 23.08.22

1)Prove that $P(x) = (x - a_1)(x - a_2) \dots (x - a_n) + p$, where p is a prime and $a_1, a_2, \dots, a_n \in \mathbb{Z}$ is irreducible in $\mathbb{Z}[x]$.

2)Given that f is a polynomial of degree 100 and $f(k) = 2^k$ for $k = 1,2,3, \dots, 101$. Find the last five digits of $f(102)$[Given that, last five digits of $f(101)$ is 10752]

3)Suppose $f(x): \mathbb{R} \to \mathbb{R}$ such that $f\big(f(x)\big) = \big(f(x)\big)^{2013}$. Show that there are infinitely many such functions satisfying this, of which exactly 4 are polynomials.

4)Prove that the positive integers n that cannot be written as a sum of r consecutive positive integers, with $r > 1$, are of the form $n = 2^l$ for some $l \geq 0$.

5) How many distinct integers are in the sequence

$$\left\lfloor \frac{1^2}{2022} \right\rfloor, \left\lfloor \frac{2^2}{2022} \right\rfloor, \left\lfloor \frac{3^2}{2022} \right\rfloor, \dots \dots, \left\lfloor \frac{2022^2}{2022} \right\rfloor$$

Assignment-5

Date: 27.9.22

1) Prove that $x^2 + px - q$ and $x^2 - px + q$ both factorize into linear factors with integral coefficients, then positive integers p and q respectively the hypotenuse and area of a right angled triangle with sides of integer length. Show further that if $x^2 + px - q = (x-\alpha)(x-\beta)$ and $x^2 - px + q = (x-\gamma)(x-\delta)$, where p, q, α, β, γ, δ are integers, then α, β, γ, δ are numerically the radii of the in-circle and three ex-circles of the triangle.

2) Let $a, b, c, d \geq 1. Prove\ that\ (1+a)(1+b)(1+c)(1+d) \leq 8abcd + 8.$

3) $a, b, c \in \mathbb{R}, such\ that\ (a+c)(a+b+c) < 0.$ Prove that $\left(\frac{b-c}{2}\right)^2 > a(a+b+c)$

4) Let x_n denotes the n^{th} non square positive integer. Then $x_1 = 2, x_2 = 3, x_3 = 5, x_4 = 6\ etc.$ For a positive real number x, denote the integer closest to it by $< x >$ If $x = m + 0.5$, where m is an integer, then $< x >= m$. Eg. $< 1.2 >= 1, < 2.8 >= 3, < 3.5 >= 3$. Show that $x_n = n + < \sqrt{n} >$

5) $a, b, c > 0, ab + bc + ca = 3.$ Show that
$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq 1.$$

Assignment-6

Date: 18.9.22

1) I have an infinite number of 1's written on a blackboard. Abir chooses 2 of the integers p and q and replaces them by $\frac{p+q}{4} = k_1$(he removes p and q and then writes $\frac{p+q}{4}$). Now he repeats the process with the number k_1 and another integer to achieve k_2, and repeats again with the number k_2 and another integer to achieve k_3 such that $k_2 = \frac{1+k_1}{4}, k_3 = \frac{1+k_2}{4}$.

Since he is an immortal, he does this again and again until he is left with only a single number. Find that number.

2) Find area enclosed by the curves $\lfloor |x| \rfloor + \lfloor |y| \rfloor = 3$.

3)Plot the following graphs:

a)x^2, x^3, $\sqrt{x}$ [in the same graph] b)$\frac{x}{|x|}$, $x \neq 0$, and 0 at $x = 0$

c)$cosec\ x$, $\sec x$

4)Find number of solutions of $e^x = x^4$ by graphical method.

5)If the graph of $y = f(x)$ is given, plot the graphs of $y = [f(x)], y = f([x]), y = [f([x])], [y] = f(x), [y] = [f(x)]$, [.] represents the greatest integer function

7)If the graph of $y = f(x)$ is given, plot the graphs of $|y| = f(x), |y| = |f(x)|, |y| = |f(|x|)|$

Assignment-7

Date: 25.10.22

1) Show that the quadratic equation $x^2 + 7x - 14(q^2 + 1) = 0 (q \in \mathbb{Z})$ has no integral root.
2) What is the minimum value of p(2) if the following conditions are followed?
   *P(x) is a polynomial of degree 17.
   *all roots of P(x) are real.
   *all coefficients are positive
   *the coefficient of x^{17} is 1
   *the product of the roots of p(x) is -1.

3) If x is a real then find the minimum value of $\sqrt{(x+6)^2 + 25} + \sqrt{(x-6)^2 + 121}$

4) Let a, b be the roots of the equation $x^2 - 10cx - 11d = 0$ $and\ those\ of\ x^2 - 10ax - 11b = 0\ are\ c, d. Then\ a + b + c + d = ?\ (a \neq b \neq c \neq d)$

<u>Assignment-8</u>

1) Starting with a positive integer M written on the board, Alice plays the following game: in each move, if x is the number on the board, she replaces it with $3x + 2$. Similarly, starting with a positive integer N written on the board, Bob plays the following game: in each move, if x is the number on the board, he replaces it with $2x + 27$. Given that Alice and Bob reach the same number after playing 4 moves each, find the smallest value of M + N .
2) Let m be the smallest positive integer such that $m^2 + (m+1)^2 + \cdots + (m+10)^2$ is the square of a positive integer n. Find $m + n$.

   [Given that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$]
3) Find the number of ordered pairs (a, b) such that $(a, b) \in \{10, 11, \ldots, 30\}$,and GCD($a, b$) + LCM($a, b$) = $a + b$.
4) Let x, y be real numbers such that $xy = 1$. Let T and t be the largest and the smallest values of the expression $\frac{(x+y)^2-(x-y)-2}{(x+y)^2+(x-y)-2}$. If T + t can be expressed in the form $\frac{m}{n}$ where m, n are nonzero integers with GCD(m, n) = 1, find the value of $m + n$.
5) Let a, b, c be reals satisfying $3ab + 2 = 6b, 3bc + 2 = 5c, 3ca + 2 = 4a$.Let $\mathbb{Q}$ denote the set of all rational numbers. Given that the product abc can take two values $\frac{r}{s} \in \mathbb{Q}$ and $\frac{t}{u} \in \mathbb{Q}$, in lowest form, find r + s + t + u.

6) $f(x) = \begin{cases} 0 + \beta : \lfloor x \rfloor = 0 \; mod \; 3 \\ 1 + \beta : \lfloor x \rfloor = 1 \; mod \; 3 \\ 2 + \beta : \lfloor x \rfloor = 2 \; mod \; 3 \end{cases}$ where $\lfloor x \rfloor$ denotes the greatest integer function. If $\sum_{n=1}^{\infty} \frac{f\left(3^n \sqrt{2021}\right)}{3^n} = 0$, then find value of β.

Assignment-9

1)Prove the following version of rearrangement theorem: Let $a_1, a_2, \dots \dots a_n$ and $b_1, b_2, \dots \dots b_n$ be 2 permutations of the numbers 1,2,……n. Show that

$\sum_{i=1}^{n} i(n+1-i) \leq \sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} i^2$

(Hint: Recall Cauchy-Scwartz inequality)

2)If $a, b, c > 0$ are sides of a triangle, prove that $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$

3)$a, b, c \geq 0, a + b + c \geq abc$. Prove that $a^2 + b^2 + c^2 \geq abc$

(Hint: See what happens if $a^2 + b^2 + c^2 < abc, i.e. abc > a^2$)

Assignment-10

1) In a circle C_1 with center O, let AB be a chord that is not a diameter. Let M be the midpoint of AB. Take a point T on the circle C_2 with OM as diameter. Let the tangent to C_2 at T meet C_1 in P. Show that $PA^2 + PB^2 = 4PT^2$.
2) A right-angled triangle has sides 3 units and 4 units. If the right angle is bisected, then find the distance between orthocentres of the smaller triangles.
3) Let ABCD be a quadrilateral such that AB+CD=AD+BC. Prove that the circles inscribed in ΔABC & ΔACD are tangent to each other. [Hint: Let the center of the circle inscribed in ADC is O_1, and that of ABC is O_2. Drop perpendiculars O_1E, O_1F, O_1G on AC, AD, DC respectively and O_2I, O_2J, O_2K on BC, AC, AB. Notice $AE = AF, DF = DG\ etc$. Then try to prove $2EJ = 0$. Notice that this would suffice.]
4) In a square ABCD, E is on AB & F is on BC such that <EDF=<CDF. DE=20 Cm. Calculate AE+CF.
5) Consider an acute angled triangle PQR such that C, I, O are circumcenter, incenter, orthocenter respectively. Suppose <QCR, <QIR, <QOR measured in degrees are α, β, γ. Show that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} > \frac{1}{45}$

Assignment-11

1) ABCD is any quadrilateral. P, Q are two points on AB and R, S are two points on CD, such that AP = PQ = QB, DR = RS = SC. Prove that $area(PQSR) = \frac{area(APRD)+area(QBCS)}{2}$
2) Let ΔABC be a triangle of which AB=AC. The bisector of angle ACB meet AB at M. Suppose AM+MC=BC. Prove that $< BAC = 100°$
3) Let P, Q, R, S be the midpoints of the sides AB, BC, CD, DA respectively of a rectangle ABCD. If the area of the rectangle is α, then calculate the area bounded by the straight lines AQ, BR, CS, DP in terms of α.
4) In a quadrilateral ABCD, AB=AD=13, BC=CD=20, BD=24. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r?

Assignment-12

1)Find the square roots:

1. 4+3i
2. 1+i
3. i
4. 7-24i

2)Let z and w be two non-zero complex numbers, such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$, then find the relation between z and w.

3)If $|z - iRe(z)| = |z - Im(z)|$, then prove that z lies on the bisectors of the quadrants.

4)If $z_1, z_2 \in \mathbb{C}$, $z_1^2 + z_2^2 \in \mathbb{R}$, $z_1(z_1^2 - 3z_2^2) = 2$ and $z_2(3z_1^2 - z_2^2) = 11$, then find $z_1^2 + z_2^2$

5)If ω is a cube root of unity but not equal to 1, then find the minimum value of $|a + b\omega + c\omega^2|$(where a, b and c are integers but not all equal).

6)If $a, b, c, \dots, k$ are the roots of the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0[p_1, p_2, .., p_n \in \mathbb{R}]$, then prove that

$(1 + a^2)(1 + b^2) \dots (1 + k^2) = (1 - p_2 + p_4 - \cdots)^2 + (p_1 - p_3 + p_5 - \cdots)^2$

7)Given $2^7 \cos^3\theta \sin^5\theta = a\sin 8\theta - b\sin 6\theta + c\sin 4\theta + d\sin 2\theta$. Find a, b, c, d.

8)If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, prove that $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$ and $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

9)If $(1+x)^n = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n$, prove that $p_3 + p_7 + p_{11} + \cdots = \frac{1}{2}\left\{2^{n-1} - 2^{\frac{n}{2}}\sin\frac{n\pi}{4}\right\}$

Similarly,

$p_0 + p_4 + p_8 + \cdots = 2^{\frac{n}{2}-1}\cos\frac{n\pi}{4} + 2^{n-2}$, $p_0 + p_3 + p_6 + \cdots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$, $p_1 + p_4 + p_7 + \cdots = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right)$, $p_2 + p_5 + p_8 + \cdots = \frac{1}{3}(2^n + 2\cos\frac{(n+2)\pi}{3})$

10)Let p be the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have a positive imaginary part and suppose $p = re^{i\theta}$, where $r > 0$ and $0° \leq \theta \leq 360°$, find θ.