2022 Problem Solving (Day1)

Srijan Chatterjee

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1)Let $e_1, e_2, \dots \dots e_k \in \mathbb{Z}^+ \cup \{0\}$. Let $A_k (respectively\ B_k)$ be the set of all k tuples $(f_1, f_2, \dots \dots \dots f_k)$ of integers such that $0 \leq f_i \leq e_i\ \forall i\ and\ \sum_{i=1}^{k} f_i$ is even(respectively odd). Show that $|A_k| - |B_k| = 0\ or\ 1$.

2) Let $P(x) = \sum_{i=0}^{n} a_i x^i$ and $Q(x) = \sum_{i=0}^{n} b_i x^i$ be two polynomials with integral coefficients such that $a_n - b_n$ is prime and $a_n b_0 - a_0 b_n \neq 0\ and\ a_{n-1} = b_{n-1}$. Suppose $\exists r \in Q, such\ that\ P(r) = Q(r) = 0$. Prove that $r \in \mathbb{Z}$.

3) We call a number $t \in \mathbb{R}$ good if the following conditions hold:
when we have a, b, c, d $\in \mathbb{R} - \{0,1\}\ such\ that\ a + b + c + d = t\ and \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = t\ \Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} + \frac{1}{1-d} = t$
then prove that
A) Prove that if t is good, t must be equal to 2.
B) 2 is good.

4) Prove that any mono variate polynomial of any degree which takes only non-negative values (for all real value of the variable) can always be expressed as sum of squares of two polynomials.

5)Let $1,2,3,4,5,6,7,8,9,11,12, \dots ..$ be the sequence of all the positive integers which do not contain the digit 0. Write $\{a_n\}$ for this sequence. By comparing with a geometric series, show that $\sum_n \frac{1}{a_n} < 90$

[**Note that**: If we remove any digit from this sequence in the same manner as of above, then the corresponding will be convergent, however $\sum \frac{1}{n}$ is not convergent.]

6)$f : [0,1] \to [0,1], f(0) = 0, f(1) = 1, f$ is continuous and differentiable in the interval (domain of f). Prove that $\exists\ \alpha < \beta$ such that $f'(\alpha)f'(\beta) = 1$.

7)$f(x) = \begin{cases} 0 + \beta : \lfloor x \rfloor = 0\ mod\ 3 \\ 1 + \beta : \lfloor x \rfloor = 1\ mod\ 3 \\ 2 + \beta : \lfloor x \rfloor = 2\ mod\ 3 \end{cases}$ where $\lfloor x \rfloor$ denotes the greatest integer function. If $\sum_{n=1}^{\infty} \frac{f(3^n\sqrt{2022})}{3^n} = 0$, then find value of β.

8)For $n \in \mathbb{N}$, define $f_n(x) = x^n + x^{n-1} + \cdots x^2 + x - 1$

a) Prove that $f_n(x) = 0$ has a unique solution. Denote this solution by a_n. Show that $\frac{1}{2} < a_n \leq 1$.

b) Prove that $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence.

c) Prove that $a_n^{n+1} - 2a_n + 1 = 0$.

d) Find the value of $\lim_{n\to\infty} a_n$.

[**Hint:** b)Observe that $f_n(a_{n+1}) = -a_{n+1}^{n+1}, f_n(a_n) = 0\ and\ f_n$ is increasing. c)Consider $(x-1)(x^n + x^{n-1} + \cdots + x^2 + x - 1)$. d)Observe that $a_n - \frac{1}{2} < \frac{a_2^{n+1}}{2}$.]

2022 Problem Solving (Day2)

Srijan Chatterjee

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1) Prove that $x^2 + px - q\ and\ x^2 - px + q$ both factorize into linear factors with integral coefficients, then positive integers p and q respectively the hypotenuse and area of a right angled triangle with sides of integer length. Show further that if $x^2 + px - q = (x - \alpha)(x - \beta)\ and\ x^2 - px + q = (x - \gamma)(x - \delta)$, where p, q, α, β, γ, δ are integers, then α, β, γ, δ are numerically the radii of the in-circle and three ex-circles of the triangle.

2) Simplify: $\frac{\sqrt{45+\sqrt{1}}+\sqrt{45+\sqrt{2}}+\cdots\ldots\ldots\sqrt{45+\sqrt{2024}}}{\sqrt{45-\sqrt{1}}+\sqrt{45-\sqrt{2}}+\cdots\ldots\ldots\sqrt{45-\sqrt{2024}}}$

3) Consider the series $50 + n^2: 51,54,59,66,75 \ldots \ldots \ldots$ If we take the greatest common divisor of 2 consecutive terms in the series, we obtain 3, 1, 1, 3………. What is the sum of all distinct elements in the 2nd series?

4) Everyday, 100 students enter a school that has 100 lockers. All the lockers are closed when they arrive.

Student 1 opens every locker.
Student 2 closes every second locker.
Student 3 changes the state of every third locker i.e., he opens if it is closed and closes it if it open.
Student 4 changes the state of every fourth locker and so on……………so that student n changes the state of every nth locker.
One day, however a few students are absent. Regardless,

those present complete the procedure and simply skip the students who are absent. For e.g. if student is absent, then nobody changes the state of every third locker. At the end of the process, it is found that only locker 1 is open and all other 99 lockers are closed.

How many students were absent that day?

5) Let P, Q, R, S be the midpoints of the sides AB, BC, CD, DA respectively of a rectangle ABCD. If the area of the rectangle is α, then calculate the area bounded by the straight lines AQ, BR, CS, DP in terms of α.

<u>2022 Problem Solving (Day3)</u>

Srijan Chatterjee

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1)$x_i's$ are positive integers. If $\sum_{i=1}^{n} 2^{x_i} = 2386$, then find the value of $n\ and\ x_i's$. A)Without using binary representation B)Using binary representation.

2) During a recent census, a man told the census taker that he had 3 children. When asked their ages, he replied -
a) "The product of their ages is 72."
b) "The sum of their ages is the same as my house number."
The census taker ran to the door and looked at the house number. "I still can't tell", she complained.
"Oh! That's right. I forgot to tell you the oldest one likes chicken."
The census taker promptly wrote down the ages of three children. So, how old are they?

3) Let ΔABC be a triangle of which AB=AC. The bisector of angle ACB meet AB at M. Suppose AM+MC=BC. Prove that

$$< BAC = 100^\circ$$

4) f(x) is a degree 4 polynomial satisfy $f(n) = \frac{1}{n}$ for n=1,2,3,4,5. If $f(0) = \frac{a}{b}$,(a and b are co-prime positive integers), then a+b =?

5) Let x_n denotes the n^{th} non square positive integer. Then $x_1 = 2, x_2 = 3, x_3 = 5, x_4 = 6\ etc$. For a positive real number x, denote the integer closest to it by <x> If x=m+0.5,

where m is an integer, then <x>=m. Eg. <1.2> = 1, <2.8> = 3, <3.5> = 3. Show that $x_n = n + < \sqrt{n} >$

6) For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let $N_1 = 2, N_2 = 3, N_3 = 5$ be the sequence of non-square positive integers. If $m^2 < N_n < (m+1)^2$ then show that $m = [\sqrt{n} + \frac{1}{2}]$

2022 Problem Solving (Day4)

Srijan Chatterjee

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1) A staircase has n steps. A man climbs either one step or two steps at a time. Prove that the number of ways he can climb up the staircase, starting from the bottom, is $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right], n \geq 1.$

2) A natural number n is said to be good if n is the sum of r consecutive positive integers, for some $r \geq 2$. Find the number of good numbers in the set $\{1,2, \ldots . .2022\}$

3) It is known that,

$$a_1-4a_2+3a_3 \geq 0$$
$$a_2-4a_3+3a_4 \geq 0$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{98}-4a_{99}+3a_{100} \geq 0$$
$$a_{99}-4a_{100}+3a_1 \geq 0$$
$$a_{100}-4a_1+3a_2 \geq 0$$

Let $a_1=1$;find the numbers $a_2, a_3, \ldots \ldots, a_{100}$.

4) Let $a_1, a_2, \ldots \ldots a_n$ and $b_1, b_2, \ldots \ldots b_n$ be 2 permutations of the numbers 1,2,……n. Show that $\sum_{i=1}^{n} i(n+1-i) \leq \sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} i^2$

5) $a, b, c \in \mathbb{R}$ and $a+b+c \neq 0, \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$. Then prove that for all odd integers n,

$$\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}$$

2022 Problem Solving (Day5)

Srijan Chatterjee

1) Rahul and Rohit just became friends with Neha and they want to know when her birthday is. Neha gives them a list of possible dates,

May 15|May 16|June 17|June 18|July 14|July 16|July 19|Aug 14|Aug 15|Aug 17

Neha then tells Rahul and Rohit separately the month and date of her birthday respectively.

Rahul: I don't know when Neha's birthday is, but I know that Rohit doesn't know also.

Rohit: At first, I don't know when Neha's birthday is, but I know now.

Rahul: Then I also know when it is.

So, when is Neha's birthday?

2) ABCD is a cyclic quadrilateral. x, y, z are distances of A from the lines BD, BC, CD respectively. Prove that $\frac{BD}{x} = \frac{BC}{y} + \frac{CD}{z}$

3) Find the last non-zero digit of 2022!

4) 10 candidates participate for Olympiad, which is organised around a table. There are 5 versions of the test and each candidate will receive exactly one version. No 2 consecutive candidates will get same version. How many ways are there to give the questions?

5) $a, b \in \mathbb{N}$ are picked randomly. What is the probability that they are co-prime? [Given : $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$]

2022 Problem Solving (Day6)

Srijan Chatterjee

1) $\sum_{i=1}^{n}\left(\left\lfloor\frac{n}{i}\right\rfloor-\left\lfloor\frac{n-1}{i}\right\rfloor\right)=A, \sum_{i=1}^{n}\left(\left\lfloor\frac{n}{i}\right\rfloor^2-\left\lfloor\frac{n-1}{i}\right\rfloor^2\right)=B$, where n is a natural number. Prove that number of divisors of n is A, and sum of divisors of n is $\frac{A+B}{2}$

2) Quadratic polynomials $P(x)$ and $Q(x)$ have leading coefficients of 2 and -2, respectively. The graphs of both polynomials pass through the two points $(16, 54)$ and $(20, 53)$. Find $P(0)+Q(0)$.

3) $N \in \mathbb{N}$ is called a good number if $\exists a_1, a_2, \dots, a_n \in \mathbb{N}$ all distinct from each other such that $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}=1$. Find all such good numbers.

4) Prove that the sum of entries of the table situated in different rows and different columns is not less than 1.

$$\begin{matrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \dots\dots\dots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & & \dots\dots\dots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & & \dots\dots\dots & \frac{1}{n+2} \\ \vdots & & & & & & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & & \dots\dots\dots & \frac{1}{2n-1} \end{matrix}$$

5) The Fibonacci sequence is defined by $a_1 = 1, a_2 = 1, a_{n+2} = a_{n+1} + a_n$. Find number of n for which $\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \cdots \frac{a_n}{2^n} > 2$

6) If you distribute the modulus sign over the content in it, it's a mistake, i.e. $|a + b| = |a| + |b|$
This is a **big mistake**, it's a **false property**!
But for how many pairs of integers (a,b) such that $-10 \leq a \leq 10 \; and -10 \leq b \leq 10$, is the above said "false" property seen to be "true"?

7) Evaluate: $\int_0^2 \left(\sqrt{1 + x^3} + \sqrt[3]{x^2 + 2x}\right) dx$

<u>2022 Problem Solving (Day7)</u>

Srijan Chatterjee

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1) You are in a dark room where a table is kept. There are 50 coins placed on a table out of which 10 coins are showing tails and 40 are showing heads. The task is to divide this set of 50 coins into two groups (not necessarily of same size) such that both groups have same number of coins showing the tails. **(Hint: First make two random group of 10 and 40 coins and then think how many heads and tails are there)**

2) Evaluate: $\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+7\sqrt{\ldots\ldots\ldots\ldots\infty}}}}}$

Inventor of this series is **Ramanujan.**

3) There are two triangles $\Delta ABC\ and\ \Delta DEF, such\ that <A = <D, AB = DE = 17, BC = EF = 10, AC - DF = 12, AC + DF = ?$

4) Consider all non-empty subsets of the set {1,2,…,n}. For every such subset, we find the product of reciprocals of each of its elements. Denote the sum of all these products by S_n. For example, $S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1.2} + \frac{1}{1.3} + \frac{1}{2.3} + \frac{1}{1.2.3}$

A)Show that $S_n = \frac{1}{n} + \left(1 + \frac{1}{n}\right) S_{n-1}$.

B)Prove using (A) that $S_n = n$.

C)Prove not using (A) that $S_n = n$.

5) Suppose the set $\{1,2,3,\dots\dots 1998\}$ is partitioned into disjoint pairs $\{a_i, b_i\}$ $(1 \leq i \leq 999)$ in a manner that for each i, $|a_i - b_i|$ equals 1 or 6. Determine with proof, the last digit of the sum $S = |a_1 - b_1| + |a_2 - b_2| + \cdots |a_{100} - b_{100}|$.

6) Let $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n = m$ be positive integers. Denote by b_k the number of these a_i for which $a_i \geq k$. Find $\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} b_i$.

<u>2022 Problem Solving (Day8)</u>

Srijan Chatterjee

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1) If $S(n) = \lim_{x \to 0} \sum_{r=1}^{n} \frac{\binom{n}{r} \sin rx \cos(n-r)x}{x.2^n}$, find S(2022).

2) Find number of positive roots of the equation, where a_i's are non-negative,

$$x^n - a_1x^{n-1} - a_2x^{n-2} - - - - - - - a_{n-1}x - a_n = 0$$

3) A right angled triangle has sides 3 units and 4 units. If the right angle is bisected, then find the distance between orthocentres of the smaller triangles.

4) Five sailors survive a shipwreck and swim to a tiny island where there is nothing but a coconut tree and a monkey. The sailors gather all the coconuts and put them in a big pile under the tree. Exhausted, they agree to go to wait until the next morning to divide up the coconuts.

At 1 o'clock, the 1st sailor divides the coconuts into 5 piles, but 1 coconut is left over. He gives that to the monkey, hides his coconuts and puts the rest of the coconuts under the tree.

At 2 o'clock, the 2nd sailor wakes up. Not realising that the 1st sailor has already taken his share, he too divides the coconuts up into 5 piles, leaving one coconut over, which he gives to the monkey. He then hides his share and puts the remainder back under the tree. At 3,4,5 o'clock, the same process is carried on. In the morning, all wake up and try to look innocent. None makes a remark about the diminished pile of coconuts and none decides to be honest and admit that they have already taken their share. Instead, they divide the pile up into 5 piles, for the 6th time, and find there is yet again one over

What is the smallest amount of coconuts that there could have been in the original pile?

5) $a, b, c \in \mathbb{R}, such\ that\ (a+c)(a+b+c) < 0$. Prove that $\left(\frac{b-c}{2}\right)^2 > a(a+b+c)$

6) $f: \mathbb{R} \to \mathbb{R}\ \forall x, y \in \mathbb{R}, (x-y) \in Q \Leftrightarrow f(x) - f(y) \in Q$. Find all such functions f.

2022 Problem Solving (Day9)

Srijan Chatterjee

1) Prove that, if f is a continuous function on the closed and bounded interval $[a, b]$, then there exists at least one number $c \in (a, b)$ for which

$$\int_a^b f(t)dt = (b-a)f(c).$$

2) Find the number of real solutions of the equation: (x-1)(x-3)………(x-2021)=(x-2)(x-4)………(x-2020)

3) $a_1, a_2, \dots \dots a_n \in \mathbb{N}, a_i > 1 \forall i = 1(1)n, \sum_{i=1}^{n} \frac{1}{a_i} = k \in \mathbb{N}. M = a_1 a_2 \dots . a_n, P(x) = M(x+1)^k - (x+a_1)(x+a_2) \dots \dots (x+a_n)$. Degree of P(x) is n[n > k]. Prove that P has no positive roots.

4) For $n \in \mathbb{N}$, define $f_n(x) = x^n + x^{n-1} + \cdots x^2 + x - 1$

a)Prove that $f_n(x) = 0$ has a unique positive solution. Denote this solution by a_n. Show that $\frac{1}{2} < a_n \leq 1$.

b)Prove that $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence.

c)Prove that $a_n^{n+1} - 2a_n + 1 = 0$.

d)Find the value of $\lim_{n\to\infty} a_n$.

[**Hint:** b)Observe that $f_n(a_{n+1}) = -a_{n+1}^{n+1}, f_n(a_n) = 0 \; and \; f_n$ is increasing. c)Consider $(x-1)(x^n + x^{n-1} + \cdots + x^2 + x - 1)$. d)Observe that $a_n - \frac{1}{2} < \frac{a_2^{n+1}}{2}$.]

5) Let $x_i > 0 \in \mathbb{R}^+ \; \forall i = 1(1)n, \sum_{i=1}^{n} x_i = 1$. Prove that $\sum_{i=1}^{n} \frac{x_i}{2-x_i} \geq \frac{n}{2n-1}$.

<u>2022 Problem Solving (Day10)</u>

Srijan Chatterjee

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1) Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers defined recursively by $a_{n+1} = \frac{3a_n}{2+a_n}, \forall n \geq 1$.

i)If $0 < a_1 < 1$, then prove that $\{a_n\}_{n\geq 1}$ is strictly increasing and $\lim_{n\to\infty} a_n = 1$.

ii)If $a_1 > 1$, then prove that $\{a_n\}$ is decreasing and $\lim_{n\to\infty} a_n = 1$.

2)Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f: \mathbb{R} \to \mathbb{R}$, satisfying $|f(x) - f(y)| = 2|x - y|$, for all $x, y \in \mathbb{R}$. Justify your answer.

3)Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}$ and for all $t \geq 0$, $f(x) = f(e^t x)$. Show that the function g defined by the equation $g(x) = \int_x^{2x} \frac{f(t)dt}{t}$ for $x > 0$ is a constant function.

4)Let f be a real valued differentiable function on the real line $\mathbb{R}$ such that $\lim_{x\to 0} \frac{f(x)}{x^2}$ exists and finite. Prove that $f'(0) = 0$.

5)Let $a_1 > a_2 > \cdots > a_r$ be positive real numbers. Compute $\lim_{n\to\infty} (a_1^n + a_2^n + \cdots + a_r^n)^{\frac{1}{n}}$, prove that limit exists(or not!). Find a counter example that if the numbers are not positive, limit may not exist.

2022 Problem Solving (Day11)

Srijan Chatterjee

1)Let $\mathbb{R}$ denote the set of real numbers. Suppose a function $f: \mathbb{R} \to \mathbb{R}$ satisfies $f\left(f\big(f(x)\big)\right) = x, \forall x \in \mathbb{R}$. Show that

a)f is one to one.

b)f cannot be strictly decreasing, and

c)if f is strictly increasing, then $f(x) = x$ for all $x \in \mathbb{R}$.

2)A real valued function f is defined on the interval $(-1,2)$.A point x_0 is said to be a fixed point of f if $f(x_0) = x_0$. Suppose that f is a differentiable function such that $f(0) > 0$ and $f(1) = 1$. Show that $f'(1) > 1$, then f has a fixed point in the interval $(0,1)$.

3)Let $f(x)$ be a continuous function, whose first and second derivatives are continuous on $[0,2\pi]$. Show that

$$\int_0^{2\pi} f(x) \cos x \, dx \geq 0$$

4)Let $g: (0, \infty) \to (0, \infty)$ be a differentiable function whose derivative is continuous, and such that $g\big(g(x)\big) = x \ \forall x > 0$. If g is not the identity function, prove that g must be strictly decreasing.

5)Let $a_0, a_1, \dots, a_{19} \in \mathbb{R}$ and

$$P(x) = x^{20} + \sum_{i=0}^{19} a_i x^i,\ x \in \mathbb{R}$$

If $P(x) = P(-x)\ \forall x \in \mathbb{R}$, and $P(k) = k^2, for\ k = 0,1,..9$.
Then find

$$\lim_{x \to 0} \frac{P(x)}{\sin^2 x}$$

2022 Problem Solving (Day12)

Srijan Chatterjee

1)Let a, b, c be three real numbers which are roots of cubic polynomial, and satisfy $a + b + c = 6, ab + bc + ca = 9$. Suppose $a < b < c$. Show that $0 < a < 1 < b < 3 < c < 4$.

2)Let $f: \mathbb{Z} \to \mathbb{Z}$ be a function satisfying $f(0) \neq 0 = f(1)$. Assume also f satisfies equations (A) and (B) below.

$$f(xy) = f(x) + f(y) - f(x)f(y) \qquad (A)$$

$$f(x - y)f(x)f(y) = f(0)f(x)f(y) \qquad (B)$$

for all integers x, y.

i) Determine explicitly the set $\{f(a) : a \in \mathbb{Z}\}$

ii) Assuming that there is a non-zero integer a such that $f(a) \neq 0$, prove that the set $\{b : f(B) \neq 0\}$ is infinite.

3)Suppose $f: [0,1] \to (0,1)$ is a continuous function. Prove that the equation

$$2x - \int_0^x f(t)\, dt = 1$$

Has exactly one solution in $(0,1)$.

4)Suppose $f \in C[0,1]$ is such that $\int_0^1 f(t)dt = 1$. Prove that there exists $c \in (0,1)$, such that $f(c) = 3c^2$.

5) Which of the polynomials $(1 + x^2 - x^3)^{1000}$ or $(1 - x^2 + x^3)^{1000}$ has the greater coefficient of x^{20} after expansion and collecting the terms?

2022 Problem Solving (Day13)

Srijan Chatterjee

1) Let $d \in \mathbb{Z}^+$. Prove that $\exists$ a right angled triangle with rational sides and area equal to d iff there exists an arithmetic progression x^2, y^2, z^2 of squares of rational numbers whose common difference is d.

2) $X_0, X_1, \alpha \in (0,1). X_{n+1} = \alpha X_n + (1-\alpha)X_{n-1}\ \forall n \geq 1.$ Prove that $\{X_n\}$ converges and find its limit.

3) Let $\{a_n\}$ be a sequence such that $a_0 = a_1 = 1, and\ \forall n \geq 1, n(n+1)a_{n+1} = n(n-1)a_n - (n-2)a_{n-1}.$
Then $\lim_{n\to\infty} a_n = ?$

4) Determine all continuous functions $f: [0,1] \to \mathbb{R}$ that satisfy $\int_0^1 f(x)\big(x - f(x)\big)dx = \frac{1}{12}$.

5) Let $f: [0, \infty) \to \mathbb{R}$ be a non-decreasing continuous function. Show that the inequality $(z - x)\int_y^z f(u)du \geq (z - y)\int_x^z f(u)du$ holds for any $0 \leq x < y < z$.

6) Consider an acute angled triangle PQR such that C, I, O are circumcentre, in-centre, orthocentre respectively. Suppose <QCR, <QIR, <QOR measured in degrees are α,β,γ. Show that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} > \frac{1}{45}$

2022 Problem Solving (Day14)

Srijan Chatterjee

1)A sequence is called an arithmetic progression of the first order if the differences of the successive terms are constant. It is called an arithmetic progression of the second order if the differences of the successive terms from an arithmetic progression of the first order. In general, for $k \geq 2$, a sequence is called an arithmetic progression of the kth order if the differences of the successive terms form an arithmetic of the $(k-1)$th order. The numbers $4,6,13,27,50,84$ are the first six terms of some order. What is its least possible order? Find a formula for the nth term of this progression.

2)If z_1, z_2, z_3 are non-zero complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$. Then prove that z_1, z_2, z_3 lie on a circle passing through the origin.

3)Let $x_1 = x_2 = 1, x_3 = 4$ & $x_{n+3} = 2x_{n+2} + 2x_{n+1} - x_n$ $\forall n \geq 1$. Prove that x_n is square $\forall n \geq 1$.

4) Let $a = \sqrt[2022]{2022}$ which is greater between 2022 and $a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}$, where a appears 2022 times.

5) $a, b, c \in \mathbb{R}^+, a + b + c = abc. Prove\ that\ \frac{1}{\sqrt{a^2+1}} + \frac{1}{\sqrt{b^2+1}} + \frac{1}{\sqrt{c^2+1}} \leq \frac{3}{2}$.

6) $f: [a, b] \rightarrow \mathbb{R}$ is a differentiable function and (b-a)>4. Prove that $\exists c \in (a, b), such\ that\ f'(c) < 1 + f(c)^2$.

2022 Problem Solving (Day15)

Srijan Chatterjee

1) Let n be a natural number. Prove that n has a (non-zero) multiple whose representation (base 10) contains only zeros and ones.

2) Let $a \; and \; b$ be positive real numbers satisfying

$$\frac{a}{b}\left(\frac{a}{b}+2\right)+\frac{b}{a}\left(\frac{b}{a}+2\right)=2022$$

Find a positive integer n such that $\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}=\sqrt{n}$

3) Let ABCD be a convex quadrilateral inscribed in a circle with AC = 7, AB = 3, CD = 5, and AD − BC = 3. Then $BD=\frac{m}{n}$ where m and n are relatively prime positive integers. Find m + n.

4) Let the positive integers a, b, c be such that $a \geq b \geq c$ and $(a^x - b^x - c^x)(x-2) > 0$ for all $x \neq 2$. Show that a, b, c are sides of a right angled triangle.

5) Let P_n denote the collection of degree n such that the polynomial and all its derivatives have integer roots.

(a)Find a polynomial in P_2 having 2 distinct roots.

(b)Find a polynomial in P_3 having at least 2 distinct roots.

(c)For any polynomial $P \; in \; P_n$, show that the arithmetic mean of all roots of P is also integer.

6) Let $f: [0,1] \to \mathbb{R}$ is differentiable function, such that $f(0) = 0$ and $f(1) = 1$.

(a) Show that there exists $x_1 \in (0,1)$ such that $\frac{1}{f'(x_1)} = 1$.

(b) Show that there exists $x_1, x_2\ distinct\ \in (0,1)$ such that

$$\frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} = 2$$

(c) Show that for a positive integer n, there exist distinct $x_1, x_2, \dots, x_n \in (0,1)$ such that

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n$$

7) Given a natural number $n > 1$, add up all the fractions $\frac{1}{pq}$, where $p\ and\ q$ are relatively prime, $0 < p < q \leq n$, and $p + q > n$. Prove that the result is always $\frac{1}{2}$.

8) Let a, b, c be positive real numbers. Prove that

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}$$

9) Prove that for any distinct integers $a_1, a_2, \dots \dots a_n$ the polynomial $P(x) = (x - a_1)(x - a_2) \dots (x - a_n) - 1$ cannot be written as a product of two non-constant polynomials with integer coefficients.

10) Given $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}, find\ \int_0^\infty \frac{\sin^4 x}{x^4} dx$.

2022 Problem Solving (Day16)

Srijan Chatterjee

1) Let P(x) and Q(x) be distinct polynomials with real coefficients such that the sum of the coefficients of each of the polynomials is S. If $P(x)^3 - Q(x)^3 = P(x^3) - Q(x^3)$,then

i)Prove that $P(x) - Q(x) = (x-1)^a R(x)$ for some integer $a \geq 1$ and a polynomial R(x) with R(1) non-zero.

ii)$S^2 = 3^{a-1}$, where a is as given in i).

2) There are two magnets kept on a table quite far such that they don't have attraction between them. Also, assume that, any side of the two magnets is attractive. Now n magnets are dropped one by one from the top such that each go either side with probability ½. Then what is the probability that at the last there are (k+1) magnets and rest (n-k+1) on the other wise?

3) Consider a real valued continuous function f satisfying $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Let

$$g(t) = \int_0^t f(x)dx \, , t \in \mathbb{R}.$$

Define $h(t) = \lim_{n\to\infty} \frac{g(t+n)}{n}$, provided the limit exists. Then prove that $h(t)$ is defined for all $t \in \mathbb{R}$ and is independent of t.

4) Let $f: (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \lim_{n\to\infty} \cos^{\mathrm{n}}\left(\frac{1}{n^x}\right)$

A) Show that f has exactly one point of discontinuity.

B) Evaluate f at its point of discontinuity

5) The polynomial $P(x) = x^n + a_1x^{n-1} + \cdots a_{n-1}x + 1$ with non-negative coefficients $a_1, a_2, \dots \dots a_{n-1}$ has n real roots. Prove that $P(2) \geq 3^n$

6) $f: : \mathbb{R} \to \mathbb{R}$ continuous and monotonic function. Given,$f(2020) = 2021. \frac{f(x)-2021}{f(y)-2021} = \frac{y}{x}$, Prove that this equation has infinitely many solutions.

7) Solve for

$x \in \mathbb{R}: 3(3x^2 - 2x + 1)^2 - 2(3x^2 - 2x + 1) + 1 = x.$

8) $x \ \in \mathbb{R}^+, \left\lfloor x \left\lfloor x \lfloor x \lfloor x \rfloor \rfloor \right\rfloor \right\rfloor = 88, then\ \lfloor 7x \rfloor = ?$

9) For all natural numbers n, let

$$A_n = \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}\ [n\ many\ radicals]$$

a) Show that for $n \geq 2, A_n = 2\sin\frac{\pi}{2^{n+1}}$

b) Evaluate the limit $\lim_{n\to\infty} 2^n A_n$

10) Let p be a prime number bigger than 5. Suppose the decimal expansion of $\frac{1}{p}$ looks like $0.\overline{a_1a_2 \dots a_r}$ where the line denotes a recurring decimal. Prove that 10^r leaves a remainder of 1 on dividing by p.

2022 Problem Solving (Day17)

Srijan Chatterjee

1) A Series is formed in the following manner:

$A(1) = 1;$

$A(n) = f(m)$ number of $f(m)$ followed by $f(m)$ number of 0; m is the number of digits of $A(n-1)$ and $f(m)$ is the remainder when m is divided by 9. Find sum of the digits of $A(30)$.

2) $x_1, x_2, \dots, x_n$ be complex numbers to satisfy the following sets of equations

$$x_1 + x_2 + \cdots + x_n = n$$
$$x_1^2 + x_2^2 + \cdots + x_n^2 = n$$
$$x_1^3 + x_2^3 + \cdots + x_n^3 = n$$
$$\vdots$$
$$x_1^n + x_2^n + \cdots + x_n^n = n$$

Then, prove that $x_i = 1 \; \forall i = 1,2, \dots, n$.

3) The diagonals of a convex quadrilateral intersect at O. What is the smallest area of this quadrilateral can have, if ΔAOB and ΔCOD has areas 4 and 9 respectively?

4) Find minimum value of

$$\sqrt{x^2 + y^2 - xy} + \sqrt{(1-x)^2 + (2-y)^2 - (1-x)(2-y)}$$

where $0 < x < y$

5) $a, b, c \in \mathbb{R}$

- If $(a+c)(a+b+c) < 0$, prove that

$$\frac{(b-c)^2}{4} > a(a+b+c) \ldots.(A)$$

- If $(a+b)(a+b+c) < 0$, prove (A)
- If $(a+b)(a+c) < 0$, prove (A)
- If $(5a+3b-c)(a+b+c) < 0$, prove (A)

6) $f(x) = \int_x^{x+1} \sin(u^2)\, du, x > 0$

- Prove that $|f(x)| < \frac{1}{x}, x > 0$ [Hint: put $u^2 = t$, use by parts and use some bound for making the inequality]
- Hence or otherwise prove that

$$\lim_{n\to\infty} f(x) = 0$$

7)Let $a > b^2, a, b \in \mathbb{R}$,

$$f(x) = \sqrt{a - x\sqrt{a + x\sqrt{a - x\sqrt{a + x\sqrt{\ldots.}}}}}, \forall x \in \mathbb{R}$$

- If $g(x) = f(x) + \frac{x}{2}$, prove that g is an even function.
- Hence or otherwise prove that,

$$\sqrt{a - b\sqrt{a + b\sqrt{a - b\sqrt{a \ldots}}}} = \sqrt{a - \frac{3b^2}{4}} - \frac{b}{2}$$

8) $S \in N$ is a finite set. If $i, j \in S (i \neq j)$, then $\frac{i+j}{\gcd(i,j)} \in S$. $|S|$ denotes the cardinality of the set S.

- Prove that, if $|S| = 1$, then it is true for all set.
- For $|S| = 2$, let a, b be the two elements of S. Prove that $\frac{a+b}{\gcd(a,b)} < \max\{a, b\}$. And hence prove that,

$$\max\{a, b\} = (\min\{a, b\})^2 - \min\{a, b\}$$

- Hence or otherwise prove that $|S|$ can't be greater than equal to 3.
- Hence find all possible sets S.

2022 Problem Solving (Day18)

Srijan Chatterjee

1)$n \in \mathbb{N}, n$ has k divisors. $1 = d_1 < \cdots < d_k = n$.

$d_{13} + d_{14} + d_{15} = n$. Find k.

2)What is the largest positive integer n such that

$$\frac{a^2}{\frac{b}{29} + \frac{c}{31}} + \frac{b^2}{\frac{c}{29} + \frac{a}{31}} + \frac{c^2}{\left(\frac{a}{29} + \frac{b}{31}\right)} \geq n(a + b + c)$$

3)$f(x)$ is a cubic polynomial $x^3 + ax^2 + bx + c$ such that $f(x) = 0$ has three distinct integral roots and $f\left(g(x)\right) = 0$ doesn't have real roots, where $g(x) = x^2 + 2x - 5$. Then find minimum value of $a + b + c$.

4) Find the least possible value of $a + b$, where a, b are positive integers such that 11 divides $a + 13b$ and 13 divides $a + 11b$.

5)Find area enclosed by the curves $[|x|] + [|y|] = k$(k is a positive integer)

6)$x_1, x_2, \ldots, x_n > 0, n \geq 2$. Prove that

$$\frac{1 + x_1^2}{1 + x_1 x_2} + \frac{1 + x_2^2}{1 + x_2 x_3} + \cdots + \frac{1 + x_n^2}{1 + x_n x_1} \geq n.$$

7)There are n people in a field whose mutual distances are different(distinct). Given, n is odd and each person has a gun. After a signal, each shoot his nearest person.
A) Prove that, at least one will survive.
B) If all shoot their neighbours with lasers, prove that no 2 lasers will cross each other.

8)Suppose $a, b \in \mathbb{R}$, such that the roots of the equation $ax^3 - x^2 + bx - 1 = 0$ are all positive real numbers. Prove that
A) $0 \leq 3ab \leq 1$.
B) $b \geq \sqrt{3}$

9)$P(x) \in \mathbb{Z}(x)$ with roots 2010 and 2017 (may have other roots also). If $|P(2013)| < 10$, find $P(2013)$.

10)A positive integer is called Corona if its digits are non-decreasing from left to right (For example, 1122345 is Corona).
A) Prove that any number in the sequence $16,1156,111556,11115556, \ldots$ is Corona
B) Prove that any number in the sequence $289,27889,2778889,277788889, ..$ is Corona
C) Hence prove that $\forall n \in \mathbb{N}, \exists$ an n digit natural number which is Corona as well as perfect square also.

11)Find the number of integral solutions of $x^3 + y^4 = 7$.

[Hint: Work with modulo 3,5,7,11,13,17 and see which works]

12) In a square $ABCD, E$ is on AB & F is on BC, such that $< EDF =< CDF, DE = 20\ cm$. Calculate $AE + CF$.

13)$ABCD, CDEF, EFGH$ are squares. AF, BH intersects each other at O. Prove that $< HOF = 45°$